

Sestri Levante
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Universal Oscillations in Counting Statistics

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¹Department of Physics, Harvard University, Massachusetts

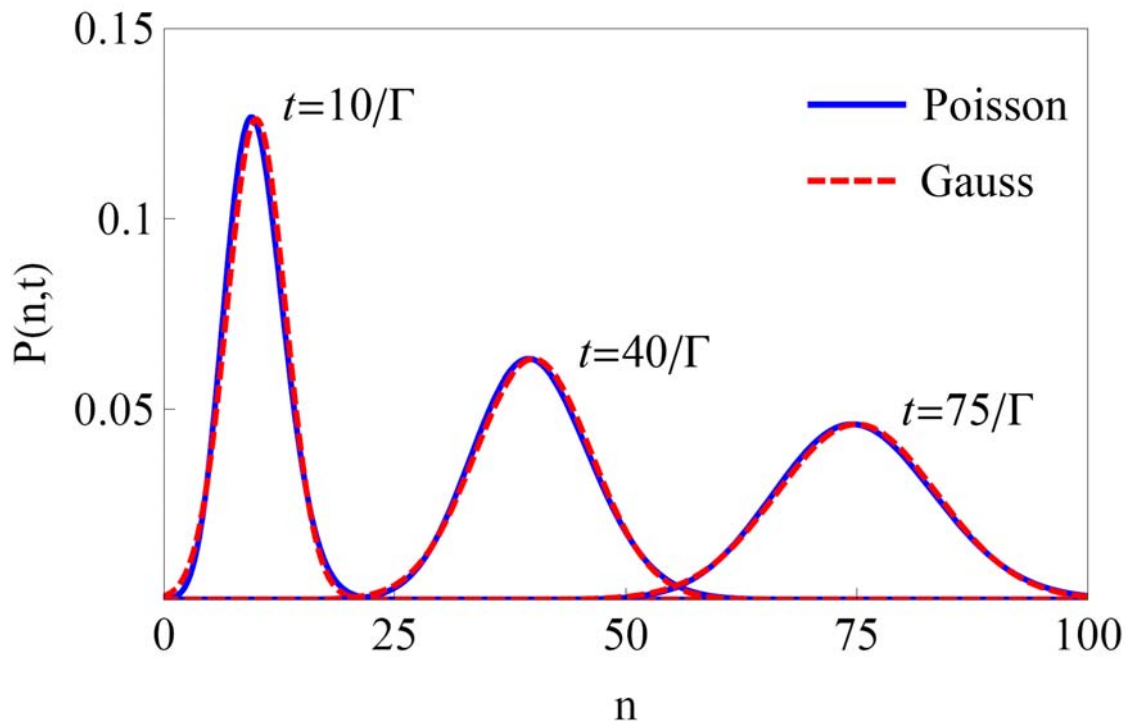
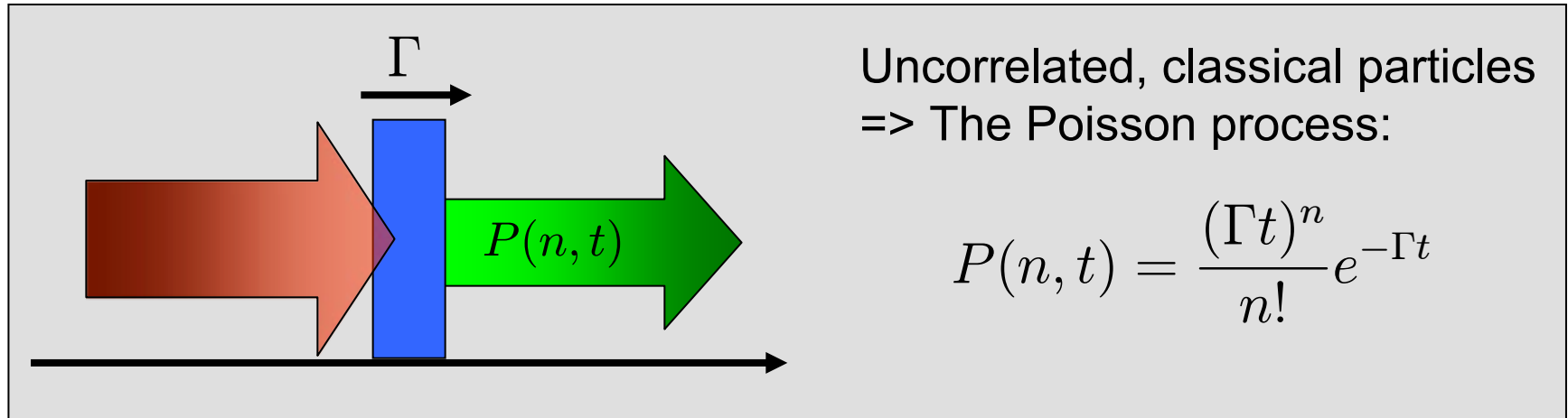
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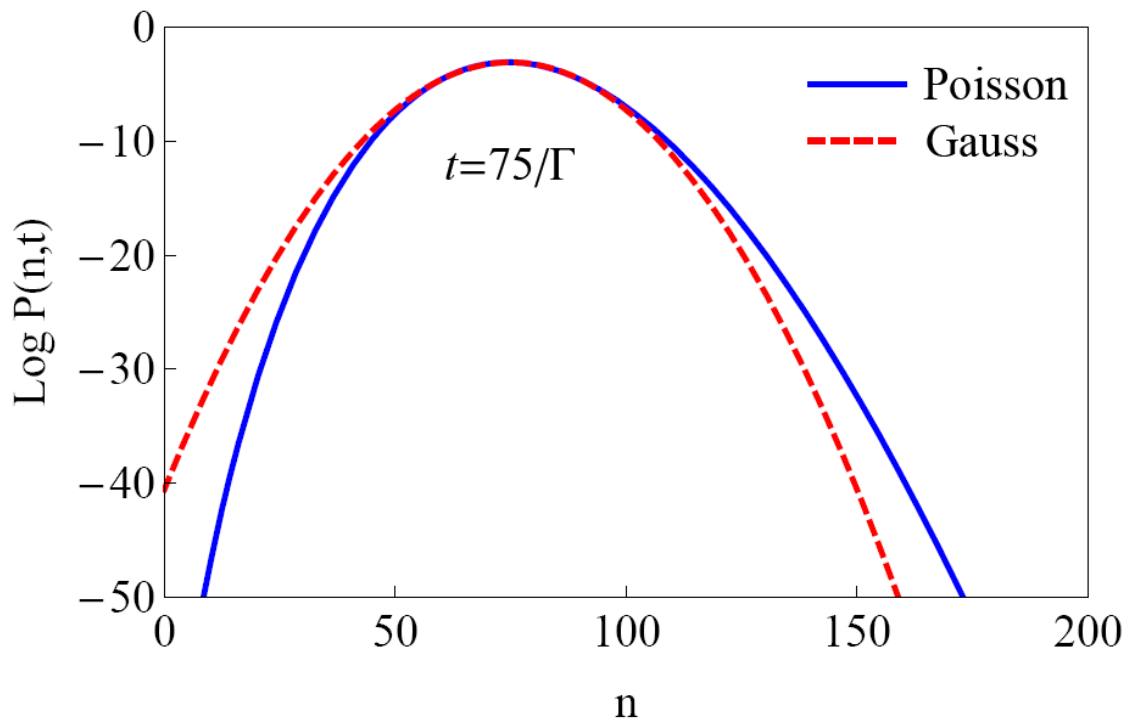
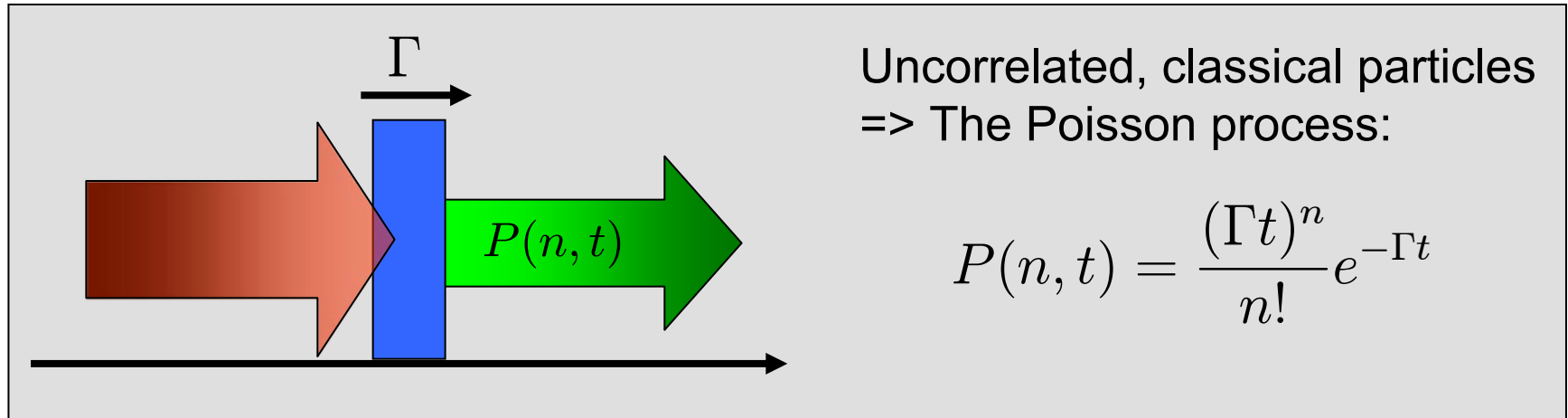
Introduction to counting statistics



Looks Gaussian...

... *but look closer!*

Introduction to counting statistics



- Peak well-described
- Tails/rare events not captured by Gaussian approximation
⇒ large deviation theory
- Consider cumulants

Introduction to counting statistics

Cumulants:	<i>Mean:</i>	$\langle\langle n \rangle\rangle = \langle n \rangle$
	<i>Variance:</i>	$\langle\langle n^2 \rangle\rangle = \langle n^2 \rangle - \langle n \rangle^2$
	<i>Skewness:</i>	$\langle\langle n^3 \rangle\rangle = \langle (n - \langle n \rangle)^3 \rangle$

Cumulant generating function (CGF): $S(z, t) \equiv \ln \sum_n P(n, t) e^{nz}$

Cumulants are *derivatives* with respect to the counting field z : $\langle\langle n^m \rangle\rangle(t) \equiv \partial_z^m S(z, t)|_{z \rightarrow 0}$

Poisson process/distribution:

$$S(z, t) = \Gamma t (e^z - 1)$$

$$\langle\langle n^m \rangle\rangle = \Gamma t, \quad m = 1, 2, 3, \dots$$

Gauss distribution:

$$S(z, t) = \mu(t)z + \sigma^2(t)z^2/2$$

$$\langle\langle n \rangle\rangle = \mu(t), \quad \langle\langle n^2 \rangle\rangle = \sigma^2(t)$$

$$\langle\langle n^m \rangle\rangle = 0, \quad m = 3, 4, 5 \dots$$

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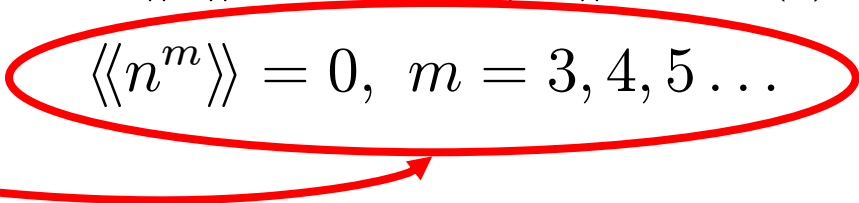
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Non-zero higher-order cumulants describe non-Gaussian behavior!



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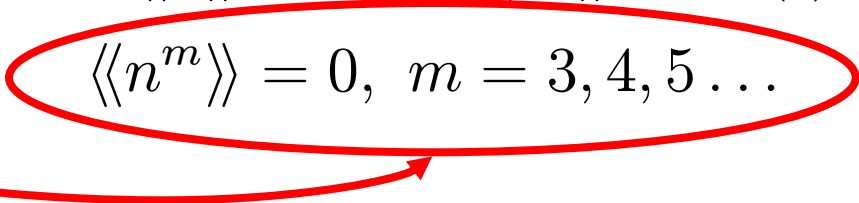
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Introduction to counting statistics

Higher-order cumulants:

- Reflect statistics beyond Gaussian approximation, e.g., rare events
- Are becoming a popular tool within mesoscopic physics to study electron transport [“Quantum Noise in Mesoscopic Physics”, Ed. Yu. V. Nazarov (2003)]
- Are believed to contain information about microscopic details and mechanisms like coherence/decoherence, entanglement, disorder...
- Are, however, generally hard to measure (good statistics needed)

Question: *What is the general behavior of high-order cumulants?*

Introduction to counting statistics

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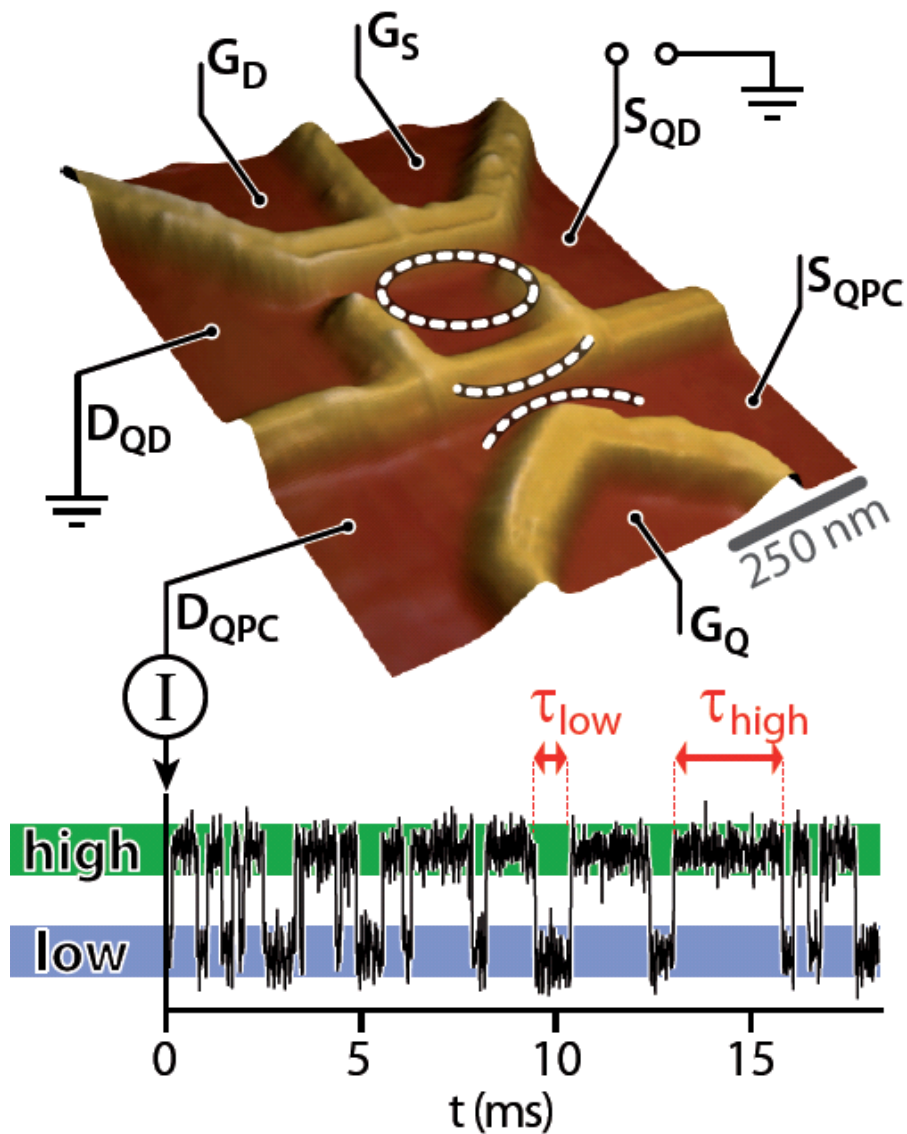
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This talk: Experiment+Theory

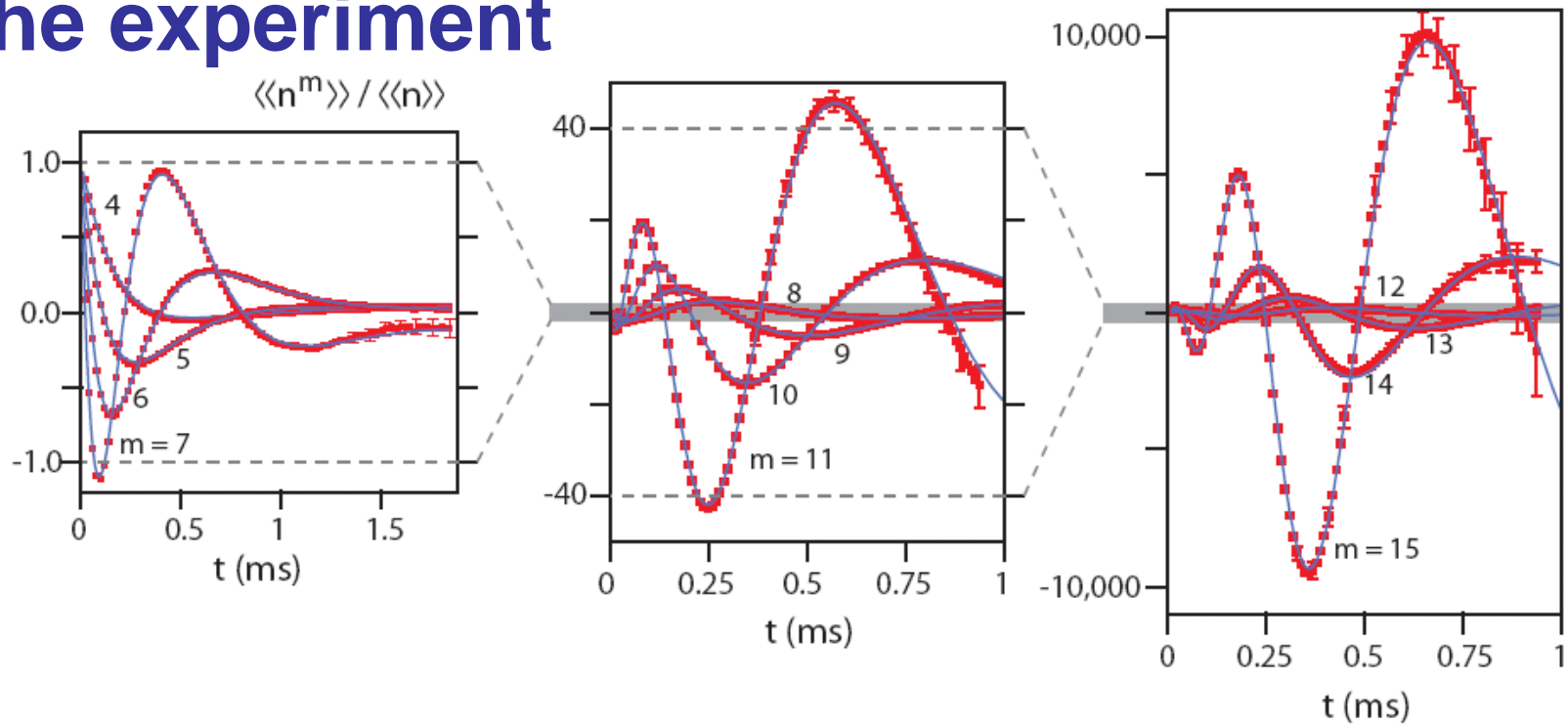
*C. Flindt, C. Fricke, F. Hohls, T. Novotný, K. Netočný, T. Brandes, and R. J. Haug, arXiv:0901.0832
-- to appear in PNAS (June, 2009)*

The experiment



- Single-electron tunneling through Coulomb-blockade quantum dot (QD)
- Large bias across QD => uni-directional electron transport
- Quantum point contact (QPC) used as real-time counter
- Electrons enter QD with rate $\Gamma_S=1.46$ kHz and leave with rate $\Gamma_D=2.97$ kHz
- Similar experiments at ETH (2006) and NTT, Tokyo (2006)

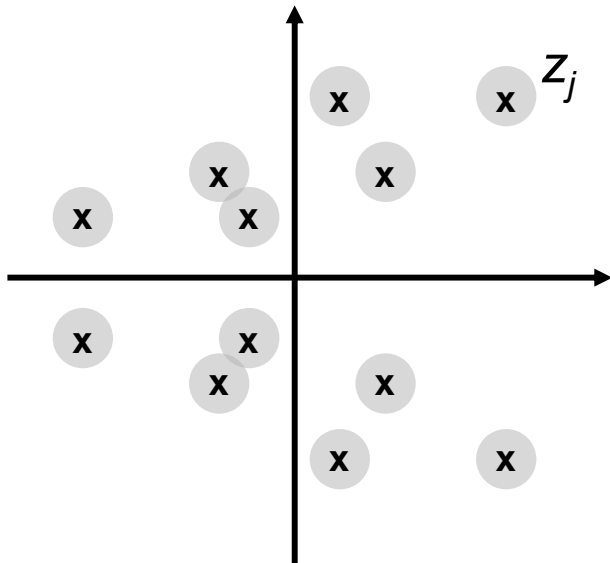
The experiment



- Experiment (dots), rate equation theory (full lines)
- The cumulants grow in magnitude with the order and oscillate with time (before reaching the linear-in-time regime around $t \sim 1.5$ ms)
- Oscillations (as functions of some parameter) found theoretically in e.g.
 - Chaotic cavities, *S. Pilgram & M. Büttiker*, PRB `03
 - Mach-Zender interferometer, *H. Förster et al.*, PRB `05
 - Double quantum dot, *C. Flindt et al*, PRL `08

General theory

- Cumulants are *derivatives* of the cumulant generating function (CGF)
- High-order derivatives of a function are determined by its singularities in the complex plane [M. V. Berry, Proc. R. Soc. A **461**, 1735 (2005)]
- Consider a general CGF $S(z, \lambda)$ with singularities at $z = z_j, j=1, 2, 3, \dots$. Here λ denotes all relevant system parameters, including time (entire functions are excluded – remember the Poisson process)



Close to a singularity z_j we can write

$$S(z, \lambda) \simeq A_j / (z - z_j)^{\mu_j}$$

for some A_j and μ_j , and furthermore

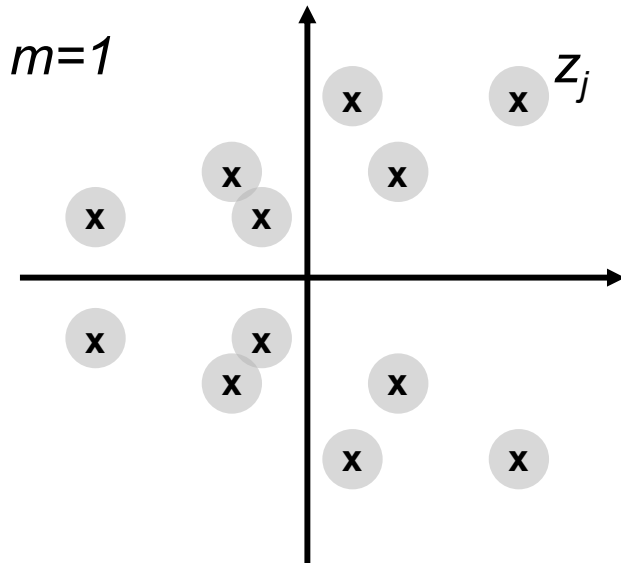
$$\partial_z^m S(z, \lambda) \simeq (-1)^m A_j B_{m, \mu_j} / (z - z_j)^{m + \mu_j}$$

with

$$B_{m, \mu_j} \equiv (m + \mu_j - 1)(m + \mu_j - 2) \dots \mu_j$$

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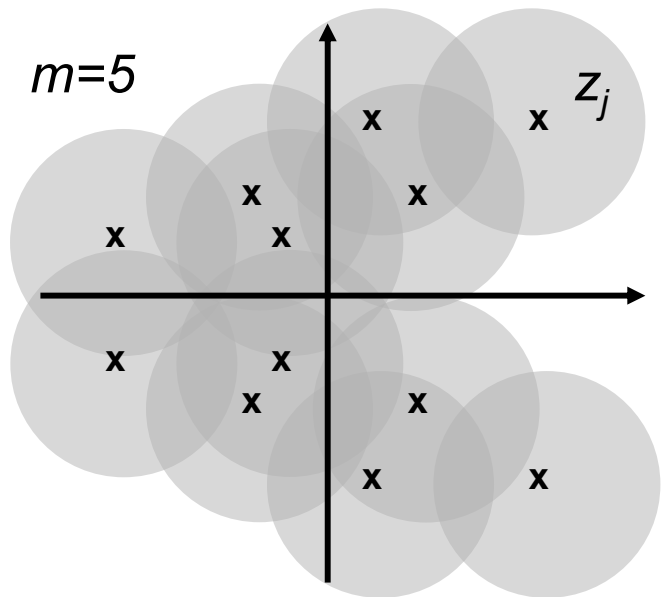
The approximation

$$\partial_z^m S(z, \lambda) \simeq (-1)^m A_j B_{m, \mu_j} / (z - z_j)^{m + \mu_j}$$

becomes increasingly better away from z_j as m is increased (Darboux Theorem).

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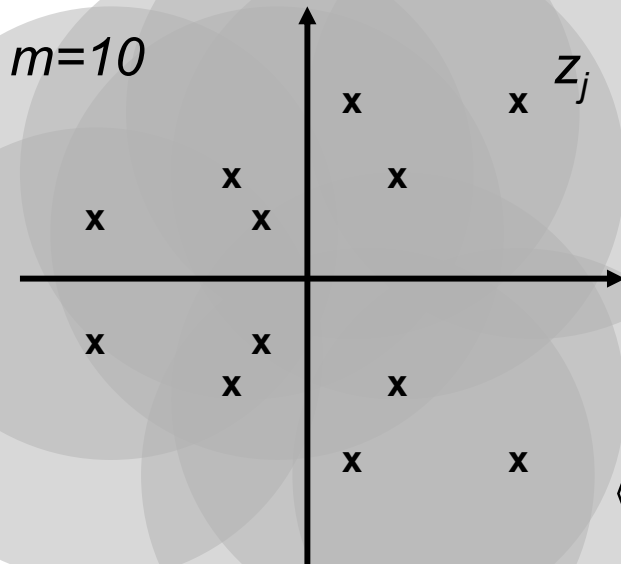
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We thus find

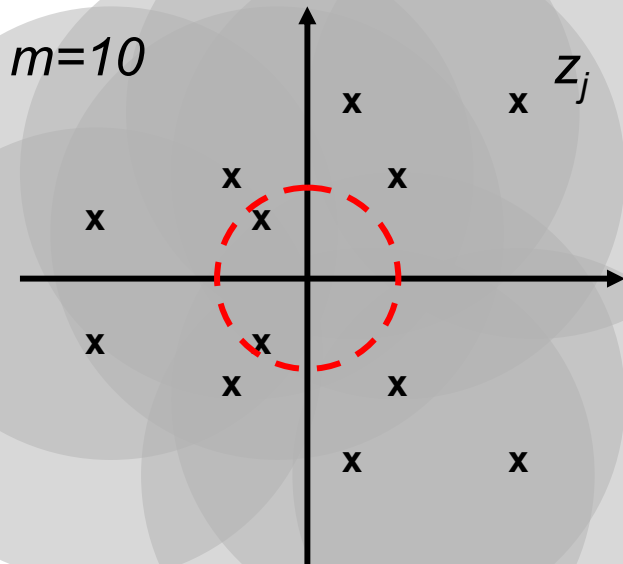
$$\langle\langle n^m \rangle\rangle = \partial_z^m S(z, \lambda) |_{z \rightarrow 0} \rightarrow \sum_j \frac{(-1)^{\mu_j} A_j B_{m, \mu_j}}{z_j^{m + \mu_j}}$$

General theory

The general result

$$\langle\langle n^m \rangle\rangle \simeq \sum_j \frac{(-1)^{\mu_j} A_j B_{m,\mu_j}}{z_j^{m+\mu_j}} = \sum_j \frac{(-1)^{\mu_j} A_j B_{m,\mu_j}}{|z_j|^{m+\mu_j}} e^{-i(m+\mu_j)\phi_j}$$

shows several ubiquitous features. Here $z_j = |z_j| \exp(i\Phi_j)$.



Ubiquitous features:

- Factorial growth of cumulants due to B_{m,μ_j}
- Singularities closest to 0 dominate the sum
- The cumulants become oscillatory functions of *any* parameter among λ that changes Φ_j (universal oscillations)

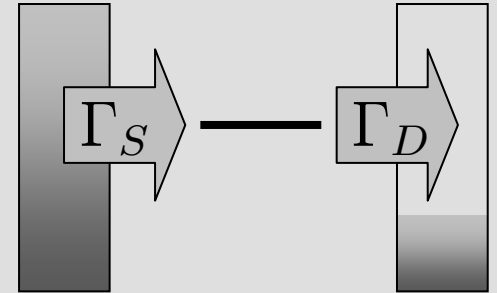
Theory behind experiment

For single-electron transport through a QD,
the CGF reads (here with ideal detector)

$$S(z, \lambda) = \frac{\Gamma t}{1+a} (w_{z,a} - 1) + \ln \frac{1 + q_{z,a} e^{-\frac{2w_{z,a}}{1+a} \Gamma t}}{1 + q_{z,a}}$$

Here

$$\Gamma \equiv \Gamma_S, \quad a \equiv \frac{\Gamma_S - \Gamma_D}{\Gamma_S + \Gamma_D}, \quad w_{z,a} \equiv \sqrt{(1 - a^2) e^z + a^2}, \quad q_{z,a} \equiv - \left(\frac{1 - w_{z,a}}{1 + w_{z,a}} \right)^2$$



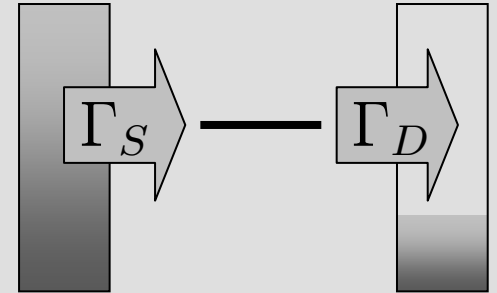
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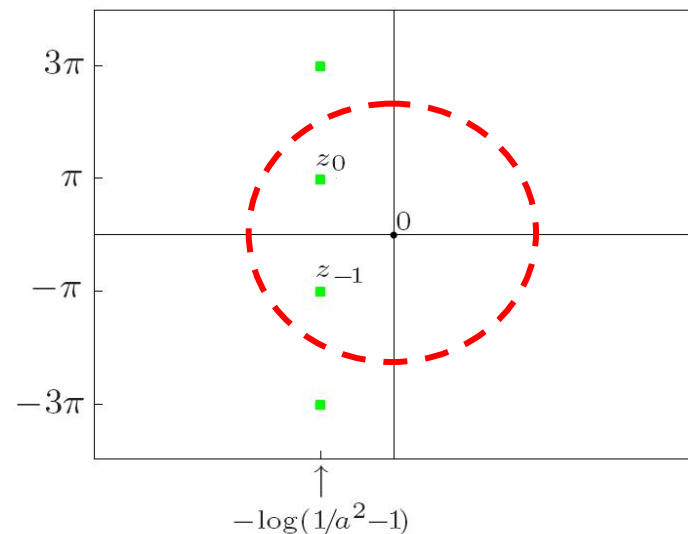


We first analyze the linear-in-time term
corresponding to the long-time limit.

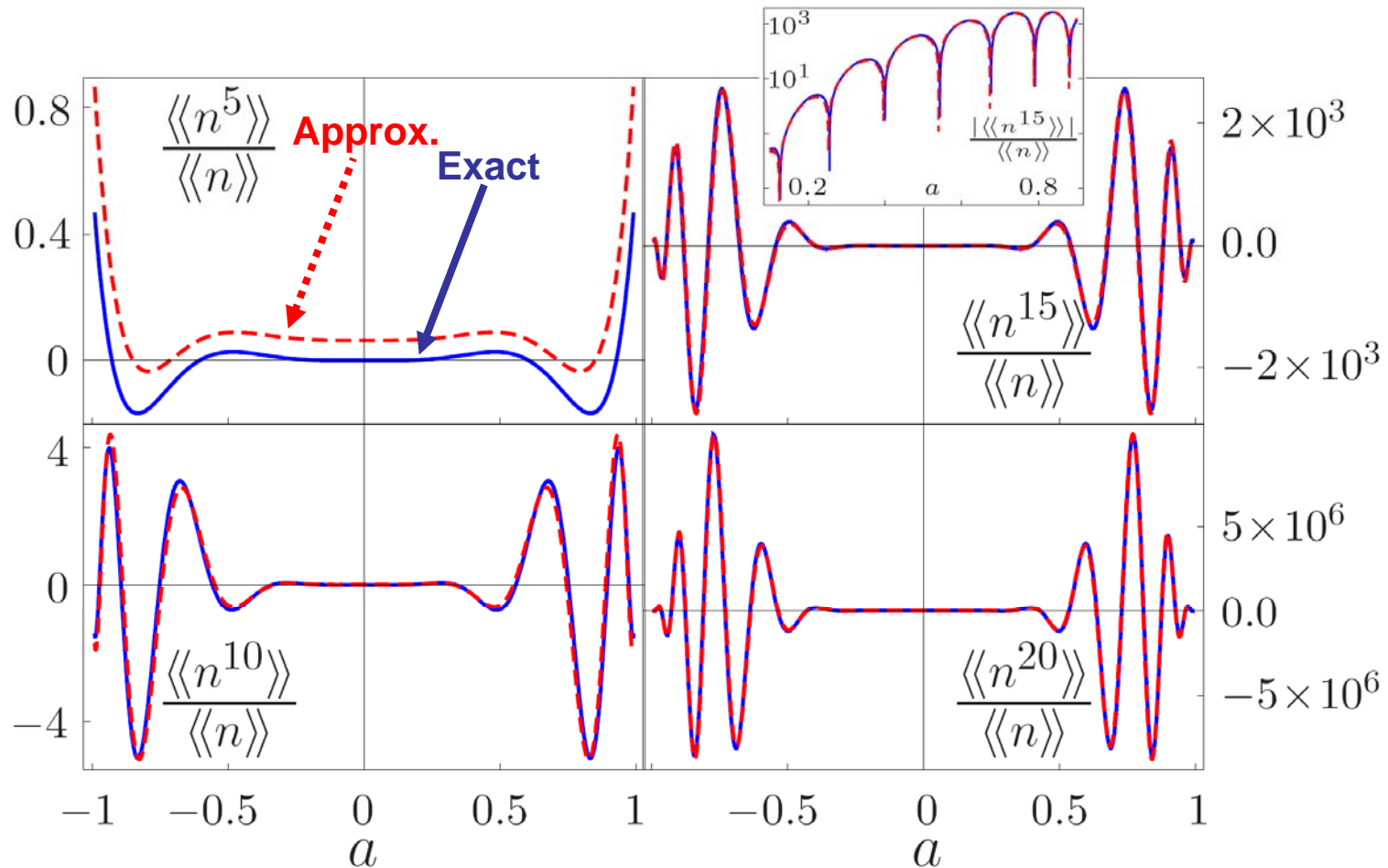
Singularities at

$$z_j = -\log(1/a^2 - 1) + (2j + 1)i\pi,$$

=> Oscillations as function of a



Theory behind experiment



Universal oscillations as function of a in the long-time limit

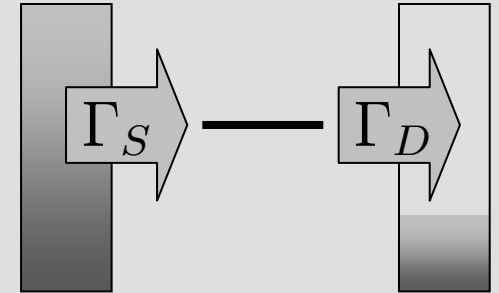
5th cumulant as function of a already measured at ETH

– *S. Gustavsson et al.*, PRB '07

Theory behind experiment

For single-electron transport through a QD, the CGF reads (here with ideal detector)

$$S(z, \lambda) = \frac{\Gamma t}{1+a} (w_{z,a} - 1) + \ln \frac{1 + q_{z,a} e^{-\frac{2w_{z,a}}{1+a} \Gamma t}}{1 + q_{z,a}}$$

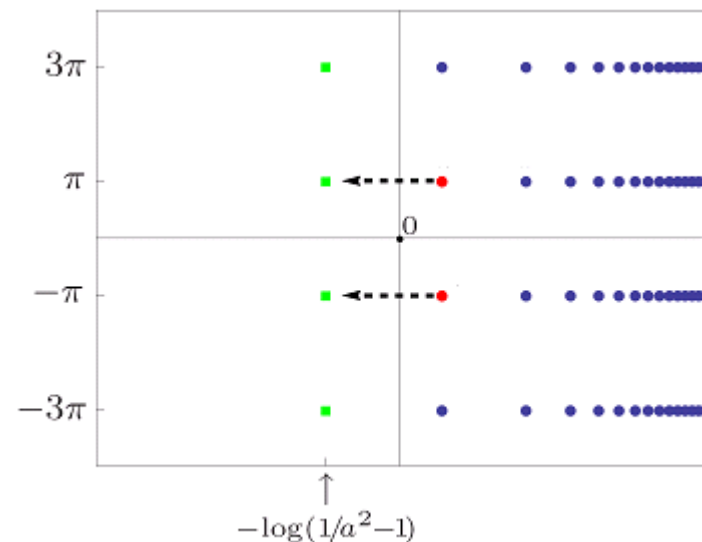


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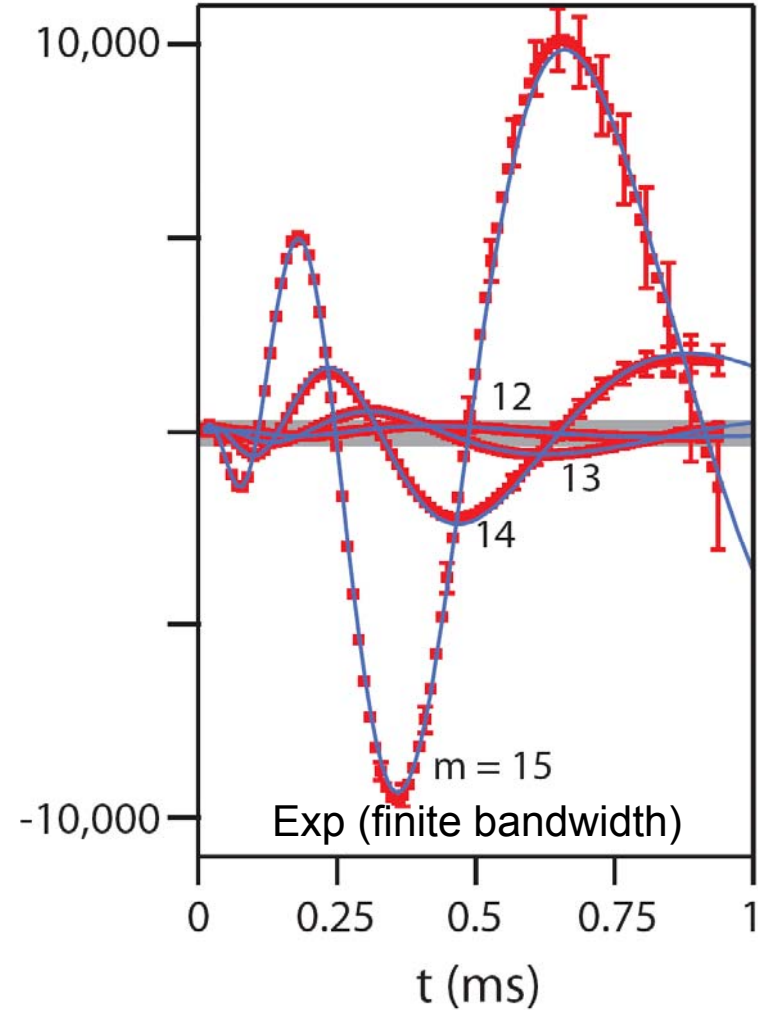
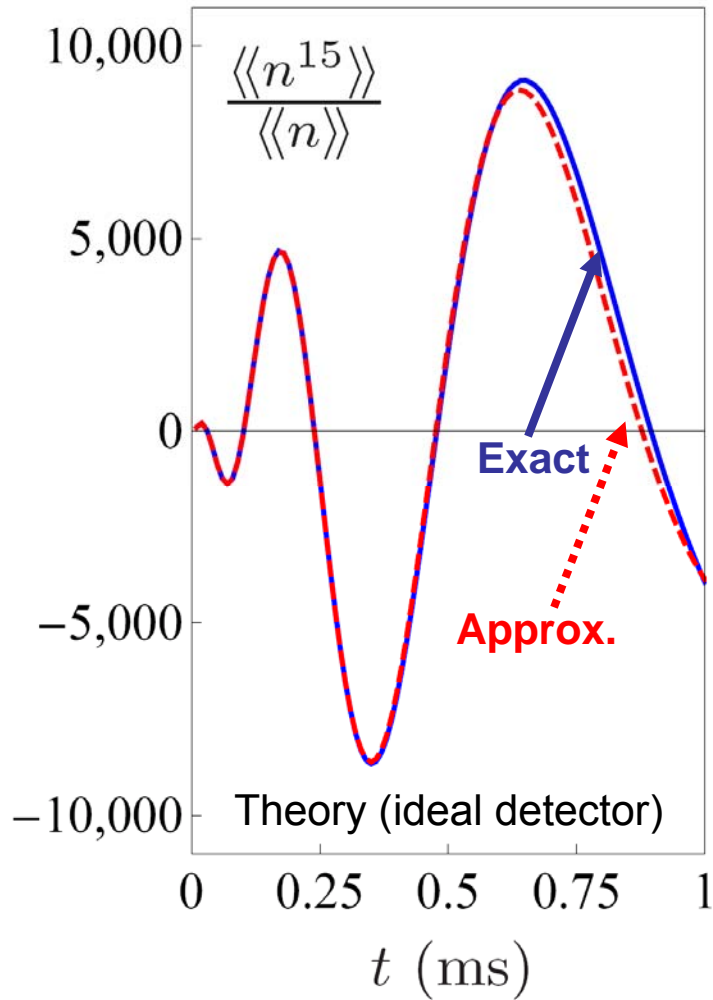
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For finite times the structure of the singularities become more involved. However, again the (time-dependent) **dominating singularities** can be found.

=> Oscillations as function of time



Theory behind experiment




Universal oscillations as function of time

Other examples

We believe that dominating singularities cause oscillations of high-order cumulants previously found theoretically in e.g.

- Chaotic cavities, *S. Pilgram & M. Büttiker*, PRB `03
- Mach-Zender interferometer, *H. Förster et al.*, PRB `05
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Similar oscillations found in other branches of physics, e.g. elementary particle physics (*I. M. Dremin et al.*, PRD `94) and quantum optics:



3 October 1994

PHYSICS LETTERS A

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Physics Letters A 193 (1994) 209–217

**Strong oscillations of cumulants of photon distribution function
in slightly squeezed states**

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Received 11 July 1994; accepted for publication 29 July 1994
Communicated by V.M. Agranovich

Conclusions

- First experimental evidence of universal oscillations in counting statistics of transport through a quantum dot
- High-order cumulants are predicted to oscillate as functions of basically any parameter as dominating singularities move in the complex plane
- Our theory explains a number of previous theoretical studies of high-order cumulants as well as our new experimental data

*C. Flindt, C. Fricke, F. Hohls, T. Novotný, K. Netočný, T. Brandes, and R. J. Haug, arXiv:0901.0832
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