

Spectral Curves and Localization in Random Non-Hermitian Tridiagonal Matrices

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Work with:

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Plan of the talk

- 1 Introduction
- 2 Spectral properties
- 3 Hatano-Nelson deformation
- 4 Spectral duality and other wonders
- 5 Conclusions

Introduction

Random tridiagonal matrices

The hermitian case is very well studied; a lot is known on:

- spectral properties
- localization of eigenvectors

one major application being the **Anderson model** ($d = 1$)

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one major application being the **Anderson model** ($d = 1$)

We made a study of **non-Hermitian tridiagonal random matrices** with corners, which find applications in

- systems with asymmetric hopping amplitudes
- 1D random walks
- models for biological systems

we consider matrices of the form

$$M = \begin{bmatrix} a_1 & b_1 & & c_1 \\ c_2 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ b_n & & c_n & a_n \end{bmatrix}$$

where the elements a, b, c are random complex numbers. We take them to be uniformly distributed in the unitary disk. We are interested in studying the **large n limit**.

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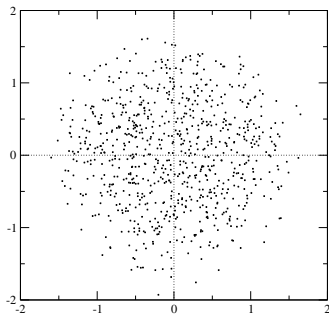
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spectrum of a sample matrix for $n = 800$

Let us look at the eigenvalue equation, which takes the form
(corners account for b.c.)

$$c_k u_{k-1} + a_k u_k + b_k u_{k+1} = E u_k$$

where E is complex; it is quite easy to show that

$$|E| \leq |a_k| + |b_k| + |c_k| \leq 3$$

however, we empirically find $|E| \leq 2$

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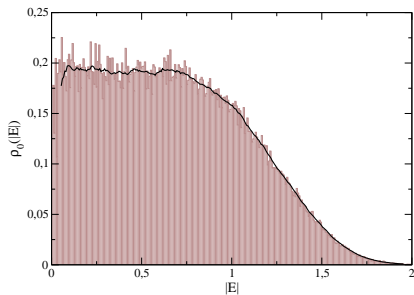
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Eigenvalue density for matrices of the M ensemble,
 $n = 1000$, 100 samples

The form of the eigenvalue equation makes it natural to introduce a 2×2 **transfer matrix**

$$t(E) = \prod_{k=1}^n \begin{bmatrix} b_k^{-1}(E - a_k) & -b_k^{-1}c_k \\ 1 & 0 \end{bmatrix}$$

such that, if

$$\Psi_n = \begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix}$$

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the two eigenvalues of $t(E)$ can be written as

$$z_{\pm}^n = e^{n(\xi_{\pm} + i\varphi_{\pm})}$$

with the interesting property that, for large n :

$$\xi_+ + \xi_- = \frac{1}{n} \log |\det t(E)| = \frac{1}{n} \sum_{k=1}^n (\log |c_k| - \log |b_k|) \rightarrow 0$$

Apply to $t(E)$ the theory of random matrix products, for large n :

- the positive exponent ξ_+ is independent of n and the realization of randomness
- and converges to the **Lyapunov exponent** of the matrix ensemble
- given by an extension of the **Thouless formula** to non-Hermitian matrices:

$$\gamma(E) = \int d^2 E' \rho_0(E') \log |E - E'| - \langle \log |b| \rangle$$

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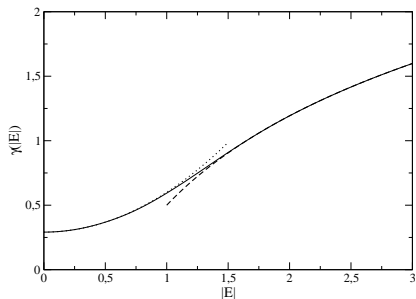
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The Lyapunov exponent is related to the **localization properties** of eigenvectors, since asymptotically

$$|u_k(E)| \simeq \exp(-\gamma(E)|k - k_0(E)|)$$

giving to $\gamma(E)$ the meaning of an inverse localization length associated to the eigenstate with eigenvalue E



Numerical evaluation of the Lyapunov exponent for the M ensemble, $n = 1000$, 100 samples

Let us look again at

$$\Psi_n = t(E)\Psi_1$$

What if we insist that Ψ_1 is an eigenvector of $t(E)$?

$$t(E)\Psi_1 \equiv z^n \Psi_1 = \Psi_n$$

this is clearly equivalent to a special deformation of the boundary conditions

$$u_{n+1} = z^n u_1, u_0 = z^{-n} u_n$$

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Following Hatano and Nelson, we introduce the following deformation of our matrices by a complex parameter $z = e^{\xi+i\varphi}$:

$$M(z^n) = \begin{bmatrix} a_1 & b_1 & & z^n c_1 \\ c_2 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ b_n/z^n & & c_n & a_n \end{bmatrix}$$

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the deformed matrix is related to

$$M_b(z) = \begin{bmatrix} a_1 & b_1/z & & zc_1 \\ zc_2 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1}/z \\ b_n/z & & zc_n & a_n \end{bmatrix}$$

by a similarity transformation, $M_b(z) = SM(z^n)S^{-1}$, through a diagonal matrix with entries $S_{ij} = z^i$.

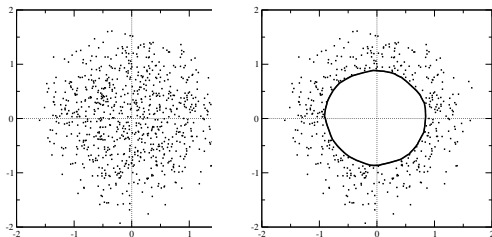
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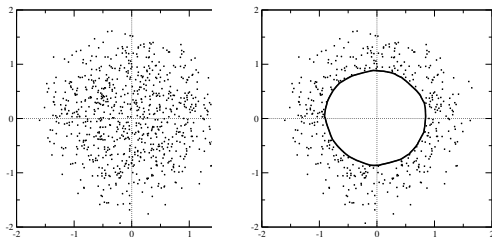
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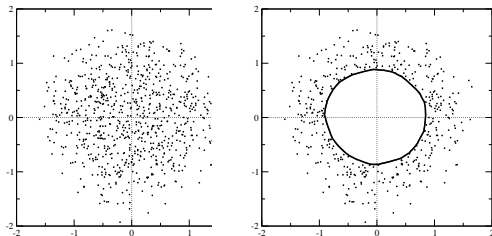
Eigenvalue spectrum with a HN deformation ($\xi = 0.5, \varphi = 0$), compared with the undeformed case ($n = 800$).

A “bubble” has appeared... The eigenvalues outside it have not moved

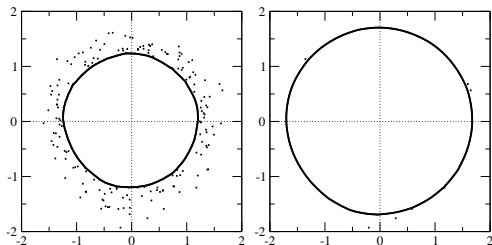


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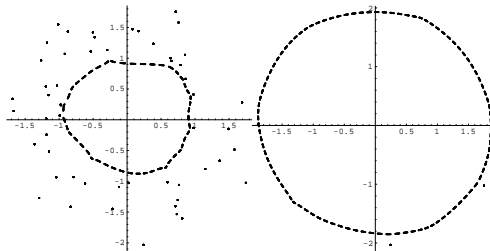
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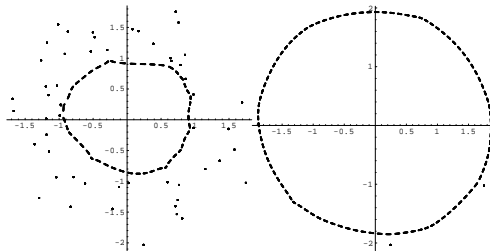
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By changing ξ to 0.7(left) and 1.0 (right) ($\varphi = 0$) the bubble grows and sweeps the entire spectrum



change φ with a step $2\pi/n$... with $\xi = 0.5$ (left) and $\xi = 1$ (right)... the eigenvalues out of the bubble do not move... while those on the bubble shift and stay on it



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We now wish to explain all these observed features.

- let $u(E)$ be an eigenstate of $M(z^n)$
- this implies that Su is an eigenstate of $M_b(z)$
- given the form of S and the localization properties of the eigenvectors u , one expects for large n :

$$|(Su)_k| \simeq \exp((\xi - \gamma(E))k)$$

implying **localization** for $\gamma(E) > \xi$ and **delocalization** for $\gamma(E) < \xi$

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$$\gamma(R) = \xi$$

we will prove this statement with

- the **Argument Principle**
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by the Argument Principle the number of zeroes of the analytic function

$$f(E) = \det(E - M_b(z))$$

inside a disk is given by the variation of

$$\frac{\arg(f(E))}{2\pi}$$

along the contour of the disk

on the other hand, the following duality identity holds

$$\det[E - M_b(z)] = -\frac{1}{z^n} (b_1 \cdots b_n) \det[t(E) - z^n]$$

we therefore have

$$\begin{aligned} \arg \det[E - M_b(e^{\xi+i\varphi})] &= \text{const.} \\ &+ \arg[e^{n(\xi_+ - \xi) + in(\varphi_+ - \varphi)} - 1] \\ &+ \arg[e^{n(\xi_- - \xi) + in(\varphi_- - \varphi)} - 1] \end{aligned}$$

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- $\xi > 0$
- $\xi_+ > 0$
- in the large n limit

We also identify $\xi_+(E)$ with the Lyapunov exponent $\gamma(|E|)$. The variation of $\arg f(E)/2\pi$ along a circumference of radius R is zero if $\xi > \gamma(R)$. \square

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- we employ a Hatano-Nelson deformation to open a hole in the spectrum of our matrices
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