

The BRST Symmetry at the Simplest Quantum Mechanical Level

Abstract

We analyse at the simplest quantum mechanical level the compensation mechanism of degrees of freedom ensured by the BRST symmetry. This analysis puts into evidence the role of a quartet of Bose-Fermi oscillators.

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Dannie Heineman Prize Talk. Denver, May 3, 2009

History

The **BRST** construction is the last episode of the long history of the **quantization of gauge fields**.

This history begins with Einstein's interpretation of the photo-electric effect based on the existence of **photons with two polarization states**.

The canonical quantization needs a **Lagrangian** function of the **four potential**, naively this implies **four "polarization" states** for the photon instead of two.

However the QED Lagrangian induces **Dirac-Bergmann first class constraints**^a associated with gauge invariance.

^asee e.g. K. Sundermeyer, *Constrained Dynamics*, Lecture Notes in Physics Vol. 169. Springer-Verlag, Berlin, Heidelberg, New-York, 1982.

In the most important situations Dirac-Bergmann's method leads to a **loss of explicit covariance and locality** which is a real disaster in view of the need of **renormalization**.

An alternative approach is based on the introduction of covariant supplementary conditions, called **gauge fixing**, reducing the number of degrees of freedom.

For a fortuitous reason, the conservation of the matter field current, in **QED the simplest choices of gauge fixing** turn out to be **dynamically stable**.

Therefore it is possible to quantize the theory by covariant and local canonical prescriptions albeit **abandoning the Hilbert space structure of the quantum state space**.

However this lucky condition does not survive the **non-abelian extensions** of QED, such as, e.g., Yang-Mills theory.

The problem has been solved by **Faddeev and Popov^a** in a **Feynman's functional integral** formulation by introducing a system of **Fermionic ghosts**.

These ghosts play the role of **compensating degrees of freedom**. The **Bosonic longitudinal and scalar components** of the gauge fields are *compensated* by the **Fermionic ghost fields**.

^aL.D.Faddeev and V.N.Popov, Phys.Letters B25 (1967) 29.

After Faddeev and Popov's paper **Slavnov and Taylor**^{ab} studied how the **Ward-Takahashi identities** of QED change in non-abelian gauge theories.

They introduced **new identities for the Green functions** which led **B.R.S** in 1974^c and, independently, **Tyutin** in 1975^d to identify the *property* that at the quantum level guarantees the **physical consistency** of the theory and, in particular, **unitarity**.

This is often called **BRST symmetry**; it is however not a physical symmetry since it acts trivially on observables.

^aJ.C.Taylor, Nucl.Phys. B33 (1971) 436.

^bA.A. Slavnov, Theor.Math.Phys.10 (1972) 99.

^cC.M.B., A.Rouet and R.Stora, Phys Letters. 52B (1974) 344.

^dL.V.Tyutin, Lebedev preprint FIAN n.39 (1975).

The Kugo-Ojima quartet mechanism

Quantum Field Theory is built on the Fock space of scattering states. This space is generated by the second quantization of the oscillators which are associated with the normal modes of the *free asymptotic fields*.

The compensating mechanism, the Kugo-Ojima quartet mechanism^a, is based on the extension to field theory of the following four oscillator model^b.

^aT. Kugo, I. Ojima, Phys.Lett.B73 (1978) 459.

^bC.M.B., C. Imbimbo (2008), Scholarpedia, 3(10):7135

http://www.scholarpedia.org/article/Becchi-Rouet-Stora-Tyutin_symmetry

Consider a **two-dimensional isotropic harmonic oscillator**. Its Hamiltonian is

$$H = \bar{P}P + \omega^2 \bar{z}z .$$

This **Bosonic oscillator** is **second quantized** through the destruction operators A and \bar{A} and their Hermitian conjugate creation operators acting on the vacuum state $|0\rangle$.

In this framework the Hamiltonian is given by

$$H = \omega(A^\dagger A + \bar{A}^\dagger \bar{A} + 1) ,$$

its spectrum $E_{(n,m)} = \omega(n + m + 1)$ with $n, m = 0, 1, 2, \dots, \infty$.

We call **Fermionic oscillator** the system with the **same Hamiltonian** and the **same equations of motion** quantized using canonical **anti-commutation relations**.

In terms of a and \bar{a} and their Hermitian conjugate creation operators its Hamiltonian is given by

$$H = \omega(a^\dagger a + \bar{a}^\dagger \bar{a} - 1) ,$$

and its spectrum $E_{(n,m)} = \omega(n + m - 1)$ with $n, m = 0, 1$.

We combine the Bosonic and the Fermionic oscillators into a super-oscillator the we call **B-F oscillator** with Hamiltonian

$$\mathbf{H} = H + \mathcal{H} = \omega(A^\dagger A + \bar{A}^\dagger \bar{A} + \bar{a}^\dagger \bar{a} + a^\dagger a) .$$

The resulting theory is invariant under **the extended super-symmetry transformations** interchanging Bosonic and Fermionic degrees of freedom generated by the **Fermionic charge**

$$Q = i(\bar{A}^\dagger \bar{a} - a^\dagger A) .$$

and its Hermitian conjugate charge

$$Q^\dagger = i(A^\dagger a - \bar{a}^\dagger \bar{A}) ,$$

Both charges are nilpotent: $Q^2 = (Q^\dagger)^2 = 0$, and satisfy the anti-commutation relation

$$\{Q, Q^\dagger\} = \frac{H}{\omega} .$$

This relation has two important consequences:

- 1- The subspace of the Fock space which is annihilated by both Q and Q^\dagger is generated by $|0\rangle$**
- 2- $\text{Ker}Q$, the subspace annihilated by Q alone, coincides with the set of vectors with the form $\alpha|0\rangle + Q|s\rangle$ for any α and $|s\rangle$.**

Our B-F oscillators appear naturally in Gauge Theories, e.g. Yang-Mills, identifying, for any momentum and color, the asymptotic longitudinal and scalar gluonic states with those of the Bosonic oscillator and those of the Faddeev-Popov ghosts with the states of the Fermionic oscillator. We want to get free of all these degrees of freedom that should be considered non-physical.

Obviously this can be done selecting as *physical* the super-symmetry invariant vector $|0\rangle$. However in field theory this turns out to be an exceedingly rigid constraint contrasting with Lorentz covariance.

If, on the contrary, we **weaken our condition** selecting $KerQ$, **too many unphysical vectors survive the selection.** These are of the **elements of ImQ** , that is the vectors of the form $Q|s\rangle$, for any $|s\rangle$.

There is however a way to overcome this difficulty **voiding ImQ of any physical content.** This requires the **interpretation of the Fock space as an *indefinite metric space*.**

The standard construction of an indefinite metric space starts from a Hilbert space \mathcal{H} and is based on the identification of a metric Hermitian operator M with vanishing kernel.^a

One defines the pseudo-inner product

$$\langle\langle s|t \rangle\rangle \equiv \langle s|M|t \rangle$$

and hence the squared pseudo-norm $\langle s|M|s \rangle$.

We normalize M by the condition

$$\langle 0|M|0 \rangle = 1$$

^aC.M.B., C. Imbimbo (2008), Scholarpedia, 3(10):7135
http://www.scholarpedia.org/article/Becchi-Rouet-Stora-Tyutin_symmetry.

If furthermore

$$Q^\dagger M = MQ$$

the elements of ImQ have zero pseudo-norm

$$\langle\langle Qs|Qs \rangle\rangle = \langle s|Q^\dagger MQ|s \rangle = \langle s|MQ^2|s \rangle = 0$$

and are orthogonal to the elements of $KerQ$.

Indeed if $Q|f \rangle = 0$

$$\langle\langle Qs|f \rangle\rangle = \langle s|Q^\dagger M|f \rangle = \langle s|MQ|f \rangle = 0 .$$

Therefore we can extend the usual probabilistic interpretation of the inner product in a Hilbert space to the pseudo-inner product in $KerQ$ since all its elements have non negative pseudo-norm.

Only the vacuum components of the elements of $KerQ$ contribute to the pseudo-norm and to the pseudo-scalar product.

Thus two vectors differing by the $Q|s\rangle$ content must be considered physically equivalent.

The new physical interpretation of the elements of $KerQ$ concerns *Q-equivalence classes of vectors*. These define a **Hilbert space** which in our simple example is one-dimensional, as wanted.

Different vectors belonging to the same equivalence class must be considered as gauge equivalent, i. e. related by a gauge transformation.

Conclusions

We have defined here the **generator Q of the BRST symmetry** on the asymptotic state space and hence on the **asymptotic fields**.

We do not discuss the **extension** of the same construction to **interacting fields**. This is the aspect of BRST symmetry commonly presented in the *semiclassical approximation* in text books and can be found e.g. in^a.

We also skip, since it is very technical, the analysis of the **renormalized BRST symmetry**^b.

^aC.M.B., C. Imbimbo (2008), Scholarpedia, 3(10):7135

http://www.scholarpedia.org/article/Becchi-Rouet-Stora-Tyutin_symmetry

^bC.M.B., A. Rouet, R. Stora, Ann. Phys. 98 (1976) 287