



Borel prescription for threshold resummation

Marco Bonvini

Dipartimento di Fisica, Università di Genova & INFN, sezione di Genova

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In collaboration with:

Stefano Forte, Giovanni Ridolfi

Plan of the talk:

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- results and comparisons

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In the threshold limit $z = \frac{Q^2}{s} \rightarrow 1$, when almost all the parton energy goes to the DY pair, these logs **need to be resummed**.

Resummation formalism

Mellin transform:
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$$\hat{\sigma}^{\text{res}}(N) = \exp \mathcal{S} \left(\bar{\alpha} \log \frac{1}{N} \right), \quad \bar{\alpha} \equiv 2\beta_0 \alpha_s(Q^2)$$

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is the Sudakov exponent: it can be computed to various log accuracies

$$\mathcal{S} \left(\bar{\alpha} \log \frac{1}{N} \right) = \underbrace{\frac{1}{\bar{\alpha}} g_1(\bar{\alpha} \log \frac{1}{N})}_{\text{LL}} + \underbrace{g_2(\bar{\alpha} \log \frac{1}{N})}_{\text{NLL}} + \underbrace{\bar{\alpha} g_3(\bar{\alpha} \log \frac{1}{N})}_{\text{NNLL}} + \dots$$

Landau pole

$$\mathcal{S}\left(\bar{\alpha} \log \frac{1}{N}\right) = \int_1^{N^2} \frac{dn}{n} g\left(\alpha_s\left(\frac{Q^2}{n}\right)\right), \quad \alpha_s\left(\frac{Q^2}{n}\right) = \frac{\alpha_s(Q^2)}{1 - \beta_0 \alpha_s(Q^2) \log n}$$

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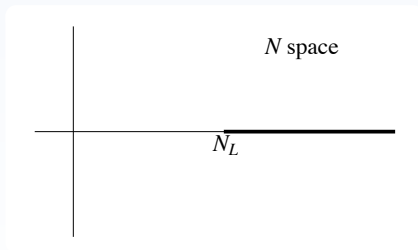
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$$N > N_L \equiv \exp \frac{1}{\bar{\alpha}}$$



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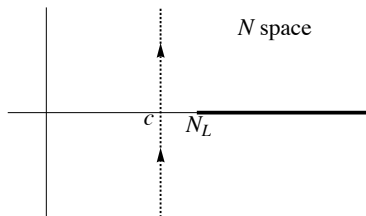
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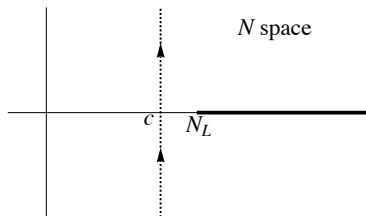
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The Mellin inverse does NOT exist

Connection with divergence of perturbative expansion

We can expand in series of $\alpha_s(Q^2)$ and invert term by term:

$$\mathcal{M}^{-1}[\gamma_{\text{LL}}] = -A \sum_{k=1}^{\infty} \frac{(-\bar{\alpha})^k}{k} \mathcal{M}^{-1} \left[\log^k \frac{1}{N} \right]$$

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A possible way out

Approximate the Mellin inversion of the single log at LL:

$$\mathcal{M}^{-1} \left[\log^k \frac{1}{N} \right] \simeq k \left[\frac{\log^{k-1}(1-z)}{1-z} \right]_+ + \text{NLL}$$

and take the sum:

$$\frac{\mathcal{M}^{-1}[\gamma_{\text{LL}}]}{A} \simeq \left[\frac{1}{1-z} \frac{\bar{\alpha}}{1 + \bar{\alpha} \log(1-z)} \right]_+ = \left[\frac{2\beta_0 \alpha_s (Q^2(1-z)^2)}{1-z} \right]_+$$

Plus distribution: $\int_0^1 dz [f(z)]_+ g(z) = \int_0^1 dz f(z) [g(z) - g(1)]$

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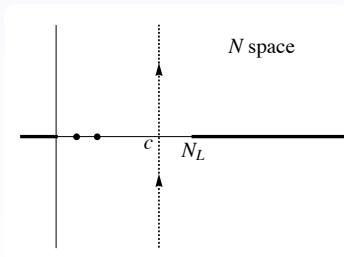
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Minimal prescription

Proposed by S.Catani, M.L.Mangano, P.Nason, L.Trentadue:

$$\sigma^{\text{MP}}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dN x^{-N} \mathcal{L}(N) \hat{\sigma}^{\text{res}}(N)$$

with $c < N_L$, as in the figure.



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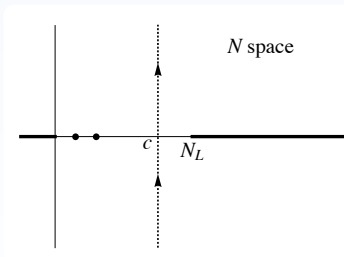
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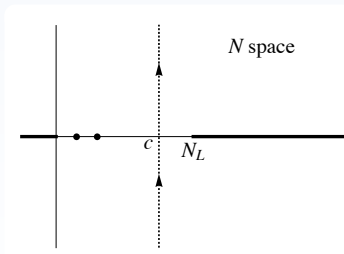
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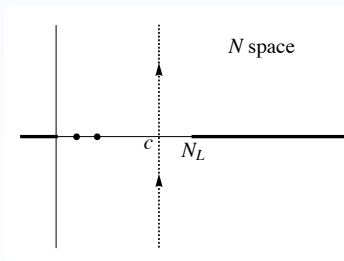
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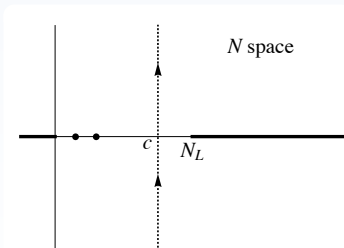
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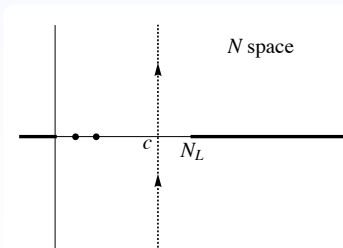
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- problems in numerical implementation



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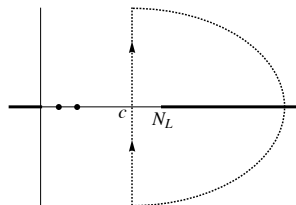
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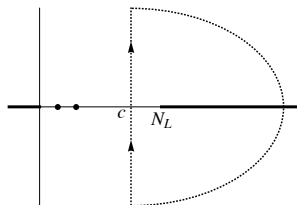
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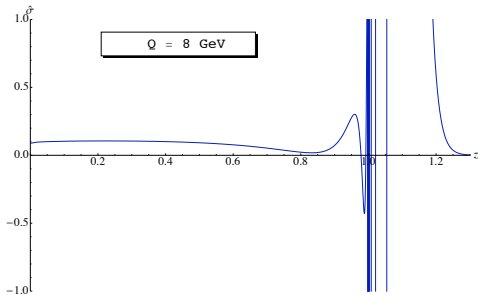
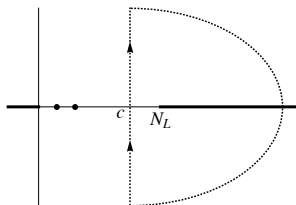


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Borel prescription (1)

Write $\exp \mathcal{S}$ as a series in its argument

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* Proposed by R.Abbate, S.Forte, G.Ridolfi, J.Rojo, M.Ubiali

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$$f_{\text{B}}(z) = \int_0^{+\infty} dw e^{-w/z} \hat{f}(w)$$

- If f_{B} exists, the series is **Borel-summable**

* Proposed by R.Abbate, S.Forte, G.Ridolfi, J.Rojo, M.Ubiali

Borel prescription (1)

Write $\exp \mathcal{S}$ as a series in its argument

$$\hat{\sigma}^{\text{res}}(N) = \exp \mathcal{S} \left(\bar{\alpha} \log \frac{1}{N} \right) = \sum_{k=0}^{\infty} h_k \left(\bar{\alpha} \log \frac{1}{N} \right)^k$$

Treat the divergent series $\mathcal{M}^{-1}(\exp \mathcal{S})$ with Borel method:*

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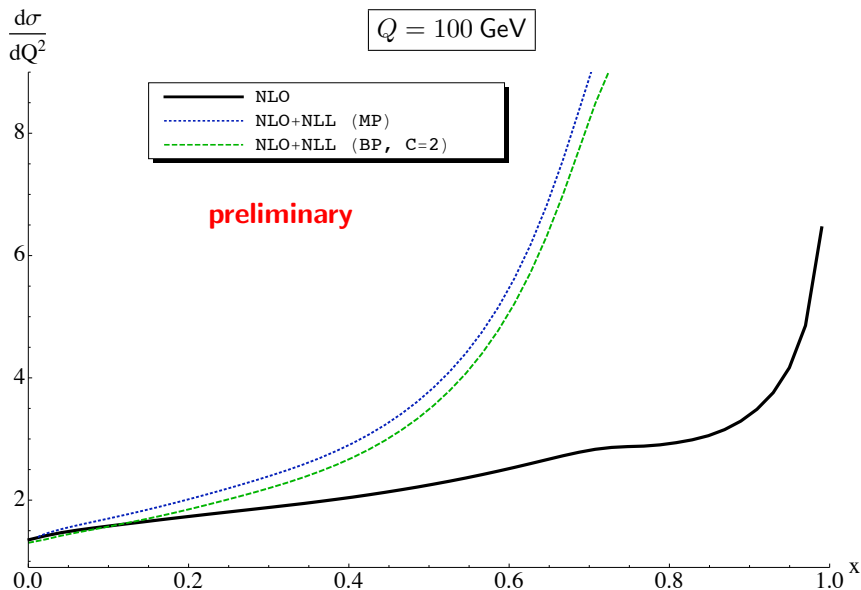
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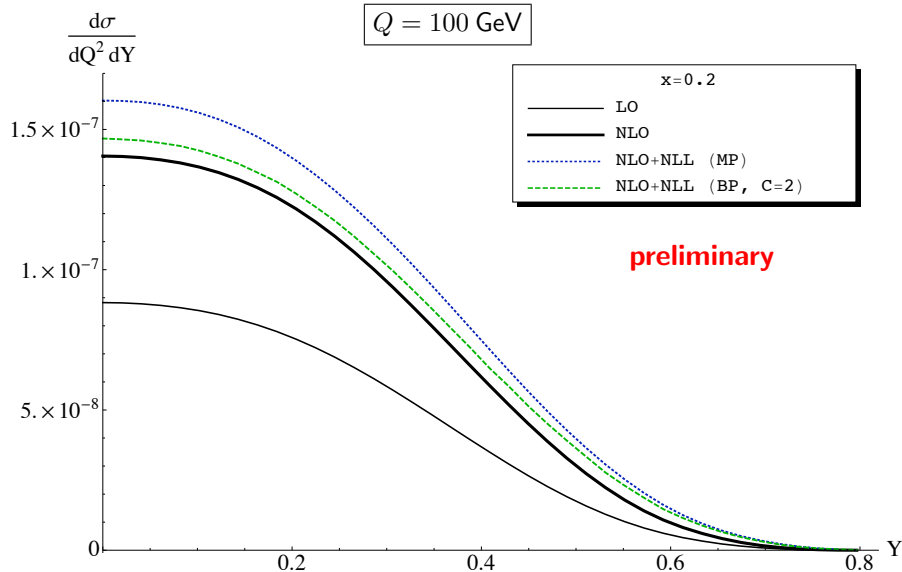
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- parameter C to estimate ambiguity
- the cut-off is related to the inclusion of higher-twist terms

$$\exp \left(-\frac{C}{\bar{\alpha}} \right) \simeq \left(\frac{\Lambda^2}{Q^2} \right)^{C/2}$$

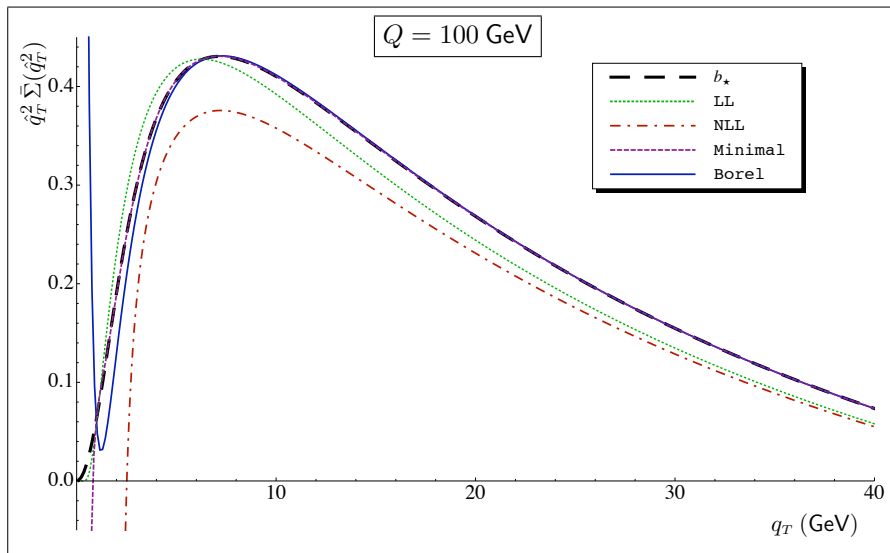
Total cross-section (pp , normalized to LO, cteq6.6 pdfs used)



Rapidity distribution (pp , cteq6.6 pdfs used)

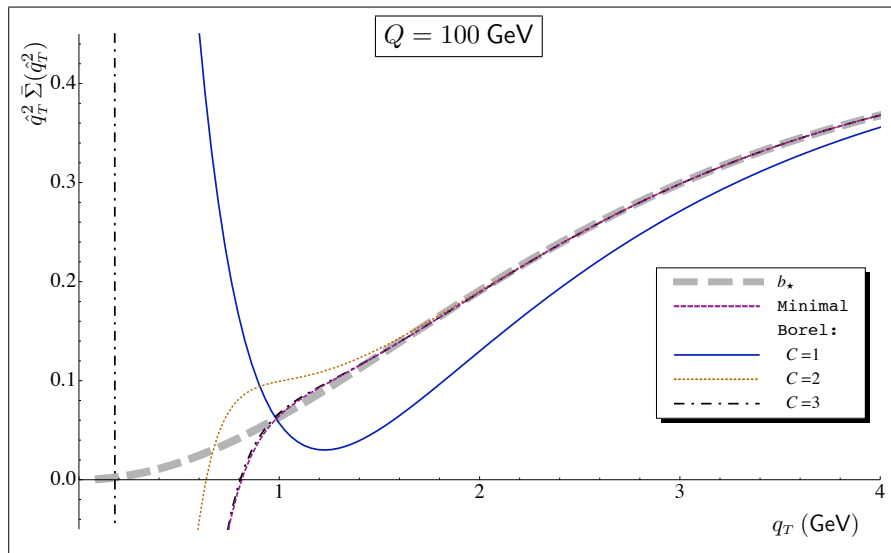


Transverse momentum distribution



M. Bonvini, S. Forte, G. Ridolfi, Nucl. Phys. B **808** (2009) 347

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- Study other processes (e.g. Higgs production)

Using Minimal prescription we get the exact inversion

$$\mathcal{M}^{-1} \left[\log \frac{1}{N} \right]_{\text{MP}} = \left[\frac{1}{\log \frac{1}{z}} \right]_+$$

Using Borel prescription we get the more physical result

$$\mathcal{M}^{-1} \left[\log \frac{1}{N} \right]_{\text{BP}} = \left[\frac{1}{1-z} \right]_+ \left(1 - e^{-\frac{C}{\alpha}} \right)$$

Deduction of Borel prescription (1)

$$\mathcal{M}^{-1} [\exp \mathcal{S} - 1] = \sum_{k=1}^{\infty} h_k \bar{\alpha}^k \mathcal{M}^{-1} \left[\log^k \frac{1}{N} \right]$$

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Then, defining $\ell = \log(1-z)$

$$\mathcal{M}^{-1} [\exp \mathcal{S} - 1] = \left[\frac{R(z)}{1-z} \right]_+$$
$$R(z) = \sum_{k=1}^{\infty} h_k \bar{\alpha}^k \sum_{j=1}^k \binom{k}{j} \Delta^{(j)}(0) \ell^{k-j}$$

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We use the identity

$$\frac{1}{2\pi i} \oint \frac{d\xi}{\xi} e^{\xi} \xi^{-(k-j)} = \begin{cases} \frac{1}{(k-j)!} & \text{for } j \leq k \\ 0 & \text{for } j > k \end{cases}$$

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