

Non-Abelian Vortices with $SO(N)$ e $USp(N)$ gauge groups

Sven Bjarke Gudnason

(Università di Pisa & INFN Sezione di Pisa)

Convegno Informale di Fisica Teorica

Sestri Levante

June 8, 2009

In collaboration with: Roberto Auzzi, Minoru Eto, Toshiaki Fujimori, Kenichi Konishi, Takayuki Nagashima, Muneto Nitta, Keisuke Ohashi and Walter Vinci.

Non-Abelian monopoles, duality and confinement

Construction of non-Abelian vortices

Eto, Fujimori, SBG, Konishi, Nagashima, Nitta, Ohashi, Vinci
arXiv:0905.3540; arXiv:0903.4471; arXiv:0809.2014; arXiv:0802.1020;

Introduction

One of our primary motivations are related to

- Confinement of quarks in Yang-Mills theory ('t Hooft-Mandelstam)
- Electric-Magnetic duality (Dirac, Montonen-Olive)
- Non-Abelian monopoles and GNO-duality (Goddard-Nuyts-Olive-Weinberg)



In relation with this emerges the assumption of “dynamical Abelianization”, which is indeed non-trivial.

- Softly broken $\mathcal{N} = 2$ pure $SU(2)$ in Seiberg-Witten theory
→ Abelian
- Generic $\mathcal{N} = 2$ super-Yang-Mills with quarks
→ non-Abelian

Seiberg 1994, Seiberg-Witten 1994, Argyres-Plesser-Seiberg 1996,

Hanany-Oz 1996 Carlino-Murayama-Konishi 2000

The question remains if magnetic monopoles of QCD are of **Abelian** or **non-Abelian** type?

Non-Abelian monopoles

Given a theory with the following symmetry-breaking pattern

$$G \rightarrow H \neq U(1) , \quad (1)$$

with the field strength tensor

$$F_{ij} \sim \epsilon_{ijk} \frac{x_k}{r^3} (\beta \cdot T_{\text{Cartan}}) . \quad (2)$$

The monopoles have the charge quantization

$$2\beta \cdot \alpha = \mathbb{Z} , \quad (3)$$

with α being the root vector of H .

The solution to this condition is taking β as **any** weight vector of the dual group \tilde{H} which has the root vectors

$$\alpha^* = \frac{\alpha}{\alpha \cdot \alpha} . \quad (4)$$

The GNOW conjecture states that the non-Abelian monopoles form a multiplet of the dual group \tilde{H} .

H	$U(N)$	$SU(N)$	$SO(2N)$	$SO(2N + 1)$
\tilde{H}	$U(N)$	$\frac{SU(N)}{\mathbb{Z}_N}$	$Spin(2N)$	$USp(2N)$

Unfortunately, there are some obstacles:

1. topological obstruction – it is not possible to define a globally well-defined set of generators isomorphic to the group H in the presence of the monopole background
2. non-normalizable zero-modes – going to zero as $r^{-\frac{1}{2}}$

⇒ we cannot quantize the non-Abelian monopole

The transformations in H and \tilde{H} , respectively are non-local. The problems arise in the Coulomb phase and in the (dual) Higgs phase. Thus, we are led to study the monopoles in the confinement phase:

$$H_{\text{Higgs phase}} \Leftrightarrow \tilde{H}_{\text{confinement phase}}$$

by electric-magnetic duality arguments.

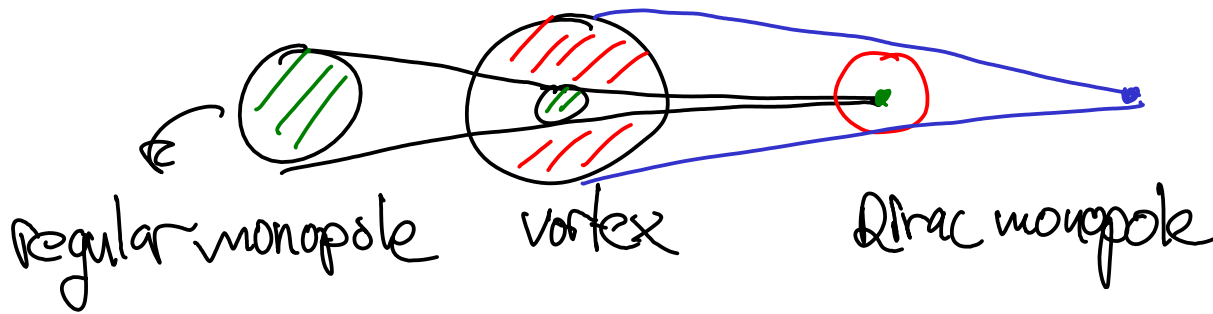
The system we are naturally led to study is

$$G \xrightarrow[\text{monopole}]{\Lambda} H \xrightarrow[\text{vortex}]{\mu} \mathbb{1}, \quad (5)$$

where we take the energy scale $\Lambda \gg \mu$.

The exact homotopy sequence for this system is

$$\dots \rightarrow \pi_2(G) \rightarrow \pi_2\left(\frac{G}{H}\right) \rightarrow \pi_1(H) \rightarrow \pi_1(G) \rightarrow \dots \quad (6)$$



As an example

we could have

$$\begin{aligned} \mathbb{1} = \pi_2(SU(3)) &\rightarrow \pi_2\left(\frac{SU(3)}{SU(2) \times U(1)}\right) \\ &\rightarrow \pi_1(SU(2) \times U(1)) \rightarrow \pi_1(SU(3)) = \mathbb{1}. \end{aligned} \quad (7)$$

In the complete system

$$G \rightarrow \mathbb{1}, \quad (8)$$

there do not exist monopoles and

$$\pi_1(G) = \mathbb{1}, \quad (9)$$

means there are also no vortices!

Consistency of the theory leads to:

Vortices confine the monopoles



Idea: vortex transformation = monopole transformation

Auzzi-Bolognesi-Evslin-Konishi 2004

Auzzi-Bolognesi-Evslin-Konishi-Yung 2003

$\mathcal{N} = 2$ super-Yang-Mills

One prime motivation lies in the fact that $\mathcal{N} = 2$ models can be studied quantum mechanically by means of the [exact Seiberg-Witten curves](#).

Much information can be gathered by studying the so-called r -vacua

$$SU(N + 1) \rightarrow U(1)^{N-r} \times U(r) \quad \text{with } N_{\text{F}} \geq 2r . \quad (10)$$

The semi-classical regime can be obtained by choosing the masses of the system as follows.

For the squarks, m and adjoint scalars, μ

$$m \gg \mu \gg \Lambda , \quad (11)$$

which in turn leads to the system

$$G \xrightarrow{m} H \xrightarrow{\sqrt{m\mu}} \mathbb{1} . \quad (12)$$

The model

As a concrete model we consider softly broken $\mathcal{N} = 2$, $SU(N+1)$ super-Yang-Mills with $N_F \geq 2N$ quark multiplets and the following superpotential

$$\mathcal{W} = \sqrt{2} \text{Tr} \left\{ Q \left(\Phi + \frac{1}{\sqrt{2}} M \right) \tilde{Q} + \mu \Phi^2 \right\}, \quad M \equiv m \mathbf{1}_{N_F}. \quad (13)$$

To obtain the wanted symmetry-breaking-pattern, we choose the following vacuum

$$\langle \Phi \rangle = \left(\begin{array}{c|c} m \mathbf{1}_N & \\ \hline & -Nm \end{array} \right), \quad (14)$$

$$\langle Q \rangle = \langle \tilde{Q}^\dagger \rangle = \left(\begin{array}{c|c} \sqrt{(N+1)m\mu} \mathbf{1}_N & \\ \hline & 0 \end{array} \right). \quad (15)$$

Keeping $m \gg \mu$, we have the symmetry breaking pattern

$$\underbrace{SU(N+1)}_{\text{monopoles}} \rightarrow \underbrace{U(N)}_{\text{vortices}} \rightarrow \mathbb{1} . \quad (16)$$

The low energy theory is of much interest to us. It is an $\mathcal{N} = 2$, $U(N)$ super-Yang-Mills theory with an FI term. We note that the gauge symmetry in this system is completely broken. However, an important remnant symmetry is the global color flavor symmetry due to the so-called color-flavor locking phase

$$q \rightarrow U_{\text{gauge}} q U_{\text{flavor}}^\dagger , \quad U_{\text{gauge}} = U_{\text{flavor}} . \quad (17)$$

This symmetry is, however, broken by the vortex configuration

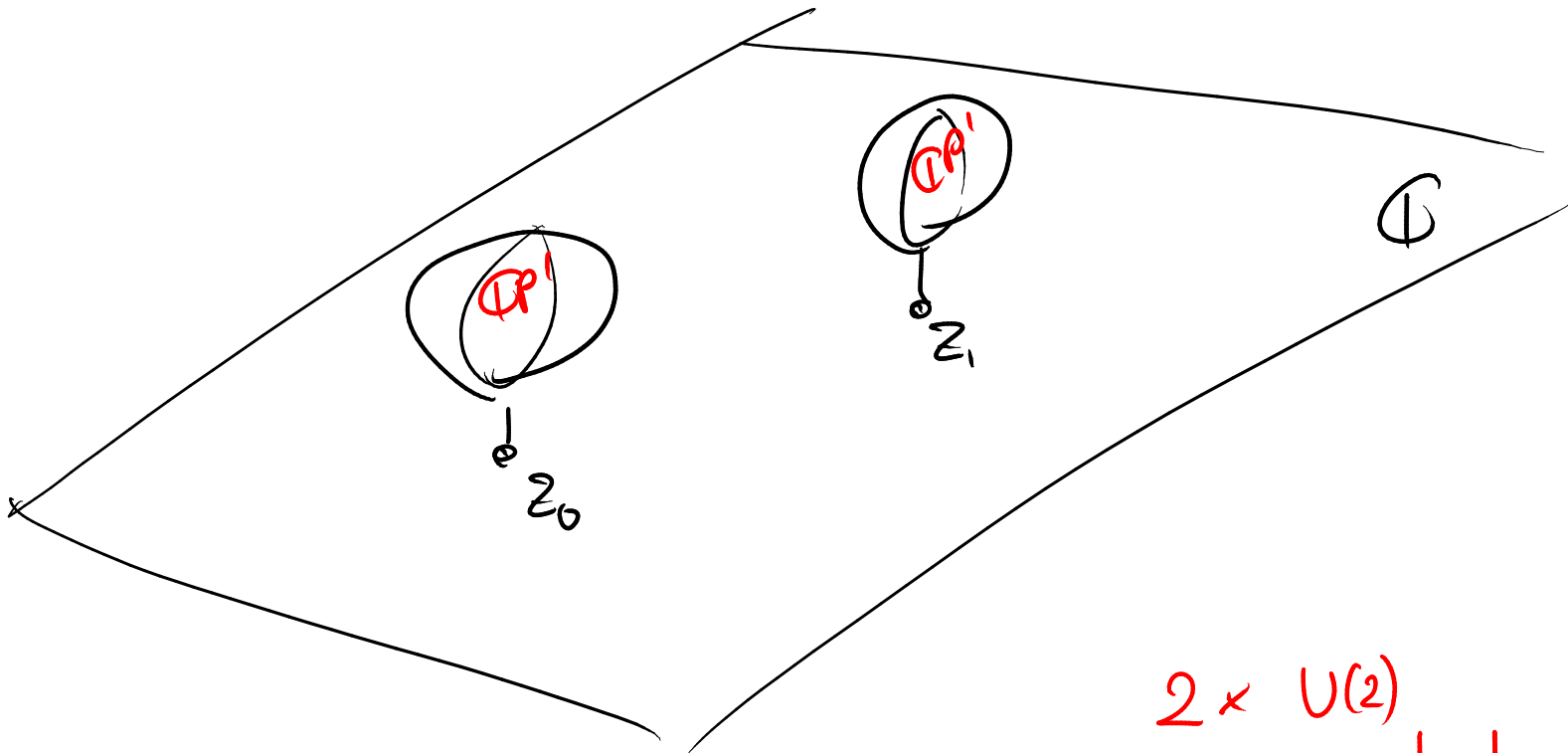
$$SU(N)_{\text{C+F}} \rightarrow U(1) \times SU(N-1) . \quad (18)$$

This in turn explains the **moduli space** of the single vortex in

$U(N)$ theory

$$\mathcal{M}_{k=1} = \mathbb{C} \times \mathbb{C}P^{N-1},$$

Hanany-Tong 2003, Auzzi-Bolognesi-Evslin-Konishi-Yung 2003, Shifman-Yung 2004



2 x U(2)
separated vortices

Summery of recent results

- In the system

$$SU(N + 1) \rightarrow U(N) \rightarrow \mathbb{1} , \quad (19)$$

The non-Abelian vortex transforms under $SU(N)_{C+F}$ in the representation \mathbf{N} while also the monopole transforms as \mathbf{N} .

- In the system $SO(5) \rightarrow U(2) \rightarrow \mathbb{1}$: the regular monopoles are confined by $k = 2$ vortices. The $k = 2$ moduli space of vortices is

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \rightarrow W\mathbb{C}P_{2,1,1}^1 ,$$

Hashimoto-Tong 2005, Auzzi-Shifman-Yung 2005

Eto-Konishi-Marmorini-Nitta-Ohashi-Vinci-Yokoi 2006

The vortex-monopole system transforms under the $3 + 1$ representation of $SU(2)$.

Weinberg 1982, Lee-Weinberg-Yi 1996, Kampmeijer-Slingerland-Schoers-Bais 2008

- A duality between 4 dim SYM and a 2 dim non-linear σ model, realized by the non-Abelian vortex

Hanany-Tong 2003, Shifman-Yung 2004

- D-brane solitons in field theory, e.g. an instanton-monopole-vortex-domain wall system

Tong 2005, Eto-Isozumi-Nitta-Ohashi-Sakai 2006, Shifman-Yung 2007

- Reconnection of cosmic strings

Hanany-Hashimoto 2004, Hashimoto-Tong 2005

Eto-Hashimoto-Marmorini-Nitta-Ohashi-Vinci 2006

- The full moduli space has been obtained

Eto-Isozumi-Nitta-Ohashi-Sakai 2005

- A model of non-Abelian vortices without dynamical Abelianization

Dorigoni-Konishi-Ohashi 2008

- A Seiberg-like duality of non-Abelian semi-local vortices

Eto-Evslin-Konishi-Marmorini-Nitta-Ohashi-Vinci-Yokoi 2007

- Non-BPS non-Abelian vortex interactions give rise to interesting distance-dependent forces

Auzzi-Eto-Vinci 2007

- The stability of non-Abelian semi-local vortices

Auzzi-Eto-SBG-Konishi-Vinci 2008

- Contrasts between confinement in sQCD and superconductors

Evslin 2008

- Multi-layer structure of non-Abelian vortices

Eto-Fujimori-Nitta-Ohashi-Sakai 2009

- Non-Abelian global vortices

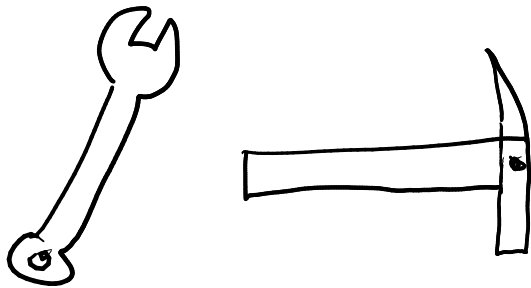
Eto-Nakano-Nitta 2009

- Non-Abelian Chern-Simons vortices with generic gauge groups.

SBG 2009

Part 2

Construction of non-Abelian vortices



The model

We will start by considering the H theory to be $\mathcal{N} = 2$ (with 8 supercharges), $U(1) \times G'$ super-Yang-Mills in 4 dimensions with N_F massless hyper multiplets.

vector multiplet : $\{V, \Phi\}$

hyper multiplets : $\{Q, \tilde{Q}\}$

Finally we will be interested in the **truncated** model, which is formally $\mathcal{N} = 1$, $U(1) \times G'$ SYM with N_F quarks Q .

$$\mathcal{L} = \text{Tr} \left[\int d^4\theta (Q^\dagger e^{-V} Q + \xi V) + \frac{1}{g^2} \int d^2\theta (WW + \text{h.c.}) \right], \quad (20)$$

where $\xi > 0$ is the FI parameter.

Component Lagrangian

Writing the model in terms of components we have

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu}^0 F^{0,\mu\nu} - \frac{1}{4g^2} F_{\mu\nu}^a F^{a,\mu\nu} + \text{Tr } \mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger - \frac{e^2}{2} \left| \text{Tr } H H^\dagger t^0 - \frac{v^2}{\sqrt{2N}} \right|^2 - \frac{g^2}{2} |\text{Tr } H H^\dagger t^a|^2, \quad (21)$$

where t^a are generators of G' .

Performing a standard Bogomol'nyi completion we arrive at the tension

$$T = -\frac{v^2}{\sqrt{2N}} \int_{\mathbb{C}} F_{12}^0 = 2\pi v^2 \nu , \quad (22)$$

together with the BPS-equations

$$\bar{D}H = 0 , \quad (23)$$

$$F_{12}^0 = e^2 \left(\text{Tr} HH^\dagger t^0 - \frac{v^2}{\sqrt{2N}} \right) , \quad (24)$$

$$F_{12}^a = g^2 \text{Tr} HH^\dagger t^a , \quad (25)$$

where ν is the topological charge.

The first equation is a “covariant holomorphic” condition, which we can solve by a **holomorphic matrix** up to complexified gauge transformations:

$$H = s^{-1}(z, \bar{z}) S'^{-1}(z, \bar{z}) H_0(z) , \quad (26)$$

with $z = x^1 + ix^2$ and the holomorphic $N \times N_F$ matrix is named the **moduli matrix**, as it encodes **all** the moduli of the vortex. The field Abelian field will have the asymptotic behavior

$$s(z, \bar{z}) \sim |z|^\nu . \quad (27)$$

To obtain the topological charge, which is the Abelian winding, we consider the holomorphic G' -invariants as

$$I_{G'}^i \left(H = s^{-1} S'^{-1} H_0 \right) = s^{-n_i} I_{G'}^i(H_0) , \quad (28)$$

together with the boundary conditions

$$I_{G'}^i(H_0) \Big|_{|z| \rightarrow \infty} = I_{\text{vev}}^i z^{\nu n_i} , \quad (29)$$

to have holomorphic polynomials of these invariants, we need to require

$$\nu n_i \in \mathbb{Z}_+ , \quad (30)$$

which we can solve by

$$\nu = \frac{k}{n_0}, \quad k \in \mathbb{Z}_+, \quad (31)$$

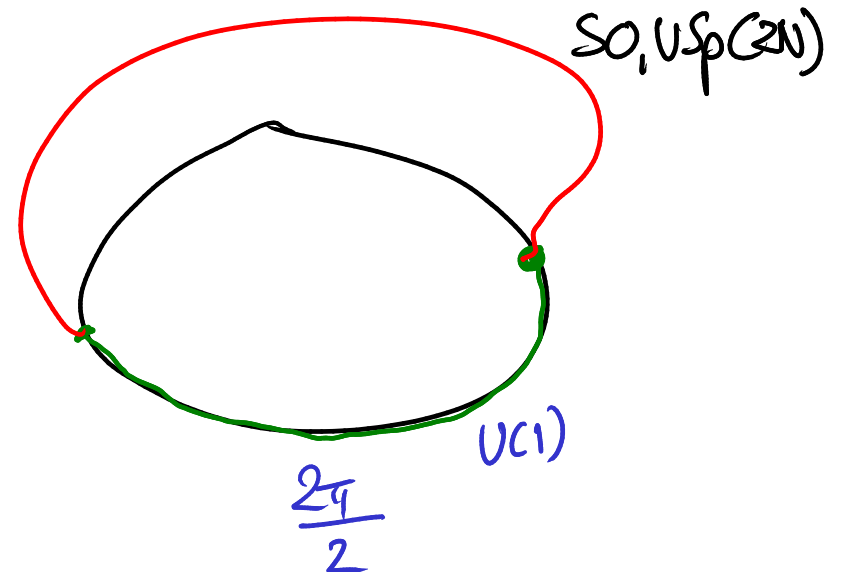
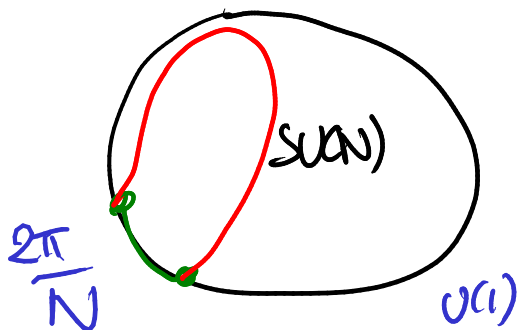
$$n_0 \equiv \text{gcd} \{n_i | I_{\text{vev}}^i \neq 0\} .$$

Thus, the **real** gauge group is

$$G = U(1) \times G' / \mathbb{Z}_{n_0} . \quad (32)$$

$$\underline{SU(N) = G'}$$

$$\underline{SO, USp(2N) = G'}$$



For some groups, we have

G	C_G	ν
$SU(N)$	\mathbb{Z}_N	k/N
$SO(2N + 1)$	1	k
$USp(2N)$	\mathbb{Z}_2	$k/2$
$SO(2N)$	\mathbb{Z}_2	$k/2$
E_6	\mathbb{Z}_3	$k/3$
E_7	\mathbb{Z}_2	$k/2$
E_8	1	k
F_4	1	k
G_2	1	k

Let us take some examples of the holomorphic invariant conditions. For $G' = SU(N)$ we have

$$\det H_0(z) = z^k + \mathcal{O}(z^{k-1}) , \quad (33)$$

For $G' = SO(2N)$ and $G' = USp(2N)$, we can build the meson field as

$$M = H^T J H, \quad (34)$$

with the

$$J_{SO(2N)} \equiv \sigma^1 \otimes \mathbf{1}_N , \quad J_{USp(2N)} \equiv i\sigma^2 \otimes \mathbf{1}_N , \quad (35)$$

and we finally end up with

$$H_0^T J H_0 = z^k J + \mathcal{O}(z^{k-1}) . \quad (36)$$

For $G' = SO(2N + 1)$ we can use

$$J_{SO(2N+1)} \equiv \begin{pmatrix} 0 & \mathbf{1}_N & 0 \\ \mathbf{1}_N & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (37)$$

and we obtain

$$H_0^\top J H_0 = z^{2k} J + \mathcal{O}(z^{2k-1}). \quad (38)$$

Let us now pursue a construction of the so-called special points of the moduli space. We have constraints on the **holomorphic invariants in terms of the moduli matrix**. By writing

$$H_0(z) = z^{\nu \mathbf{1}_N + \nu_\alpha \mathcal{H}_\alpha} \in U(1)^{\mathbb{C}} \times G'^{\mathbb{C}}, \quad (39)$$

where \mathcal{H}_α is the Cartan generators in the algebra \mathfrak{g}' while $\alpha = 1, \dots, \text{rank}(G')$ and

$$\nu = \frac{k}{n_0}.$$

For the matrix to be **single valued**, we need

$$(\nu \mathbf{1}_N + \nu_\alpha \mathcal{H}_\alpha)_{ii} \in \mathbb{Z}_{\geq 0} \quad \forall i, \quad (40)$$

which we can write as

$$\nu + \nu_\alpha \mu_\alpha^{(j)} \in \mathbb{Z}_{\geq 0} \quad \forall i, \quad (41)$$

where $\mu_\alpha^{(j)}$ are **weight vector** of the group G' .

Quantization with respect to the dual group

Subtracting two adjacent weight vectors, we arrive at the quantization condition:

$$\vec{\nu} \cdot \vec{\alpha}^{(j)} \in \mathbb{Z} ,$$

where $\alpha^{(j)}$ are root vectors of G' .

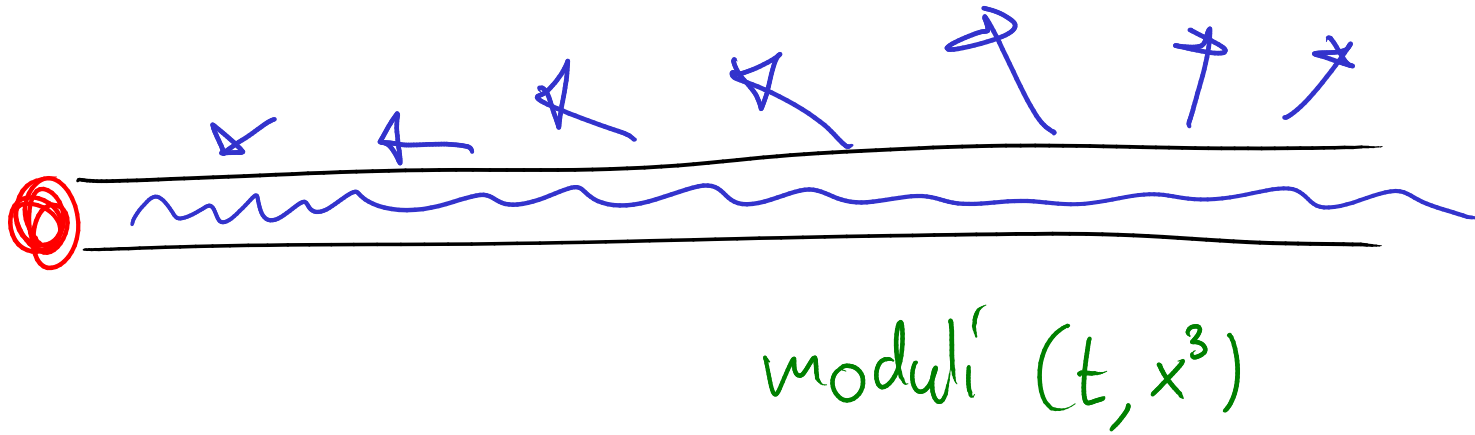
This condition is formally the Goddard-Nuyts-Olive-Weinberg quantization condition. The solution to this condition, is

$$\vec{\nu} = 2\vec{\mu} , \quad (42)$$

with $\vec{\mu}$ being any weight vector of the dual group, where the dual group has the root vectors

$$\vec{\alpha}^* = \frac{\vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}} . \quad (43)$$

An important difference from the case of the monopoles is that we have **no conceptual problems** in quantizing the zero-modes (moduli) due to an **exact color+flavor symmetry**.



\mathbb{Z}_2 parity for $G' = SO(N)$

Topological stability is provided by the first homotopy group being non-trivial

$$\pi_1 \left(\frac{U(1) \times SO(N)}{\mathbb{Z}_{n_0}} \right) = \mathbb{Z} \times \mathbb{Z}_2, \quad \begin{cases} n_0 = 1, & N \text{ odd} \\ n_0 = 2, & N \text{ even} \end{cases} \quad (44)$$

this \mathbb{Z}_2 factor is **not** present in the case of $G' = USp(2N)$. It will turn out that the moduli space is disconnected into two

$Q_+^{\mathbb{Z}_2}$

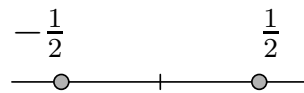
$Q_-^{\mathbb{Z}_2}$

parts of different \mathbb{Z}_2 charges.

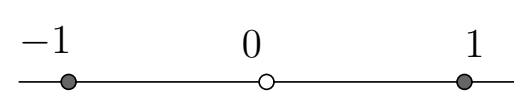
The special points for the $k = 1$ vortex with $G' = SO, USp$.



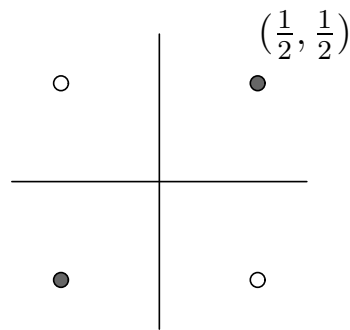
$SO(2)$



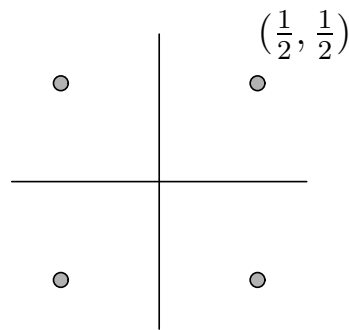
$USp(2)$



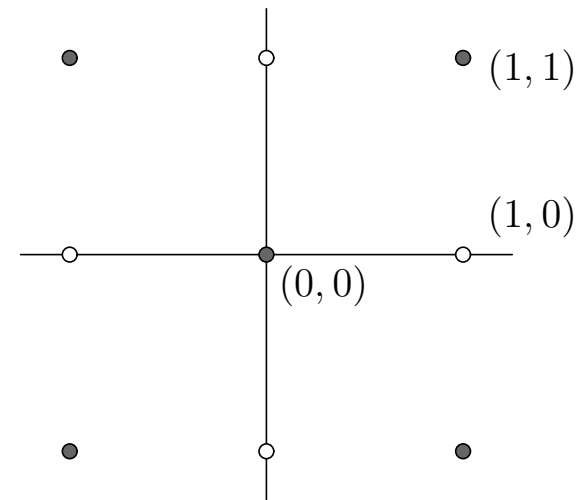
$SO(3)$



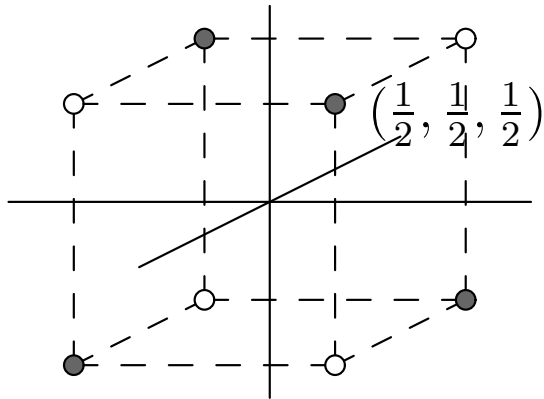
$SO(4)$



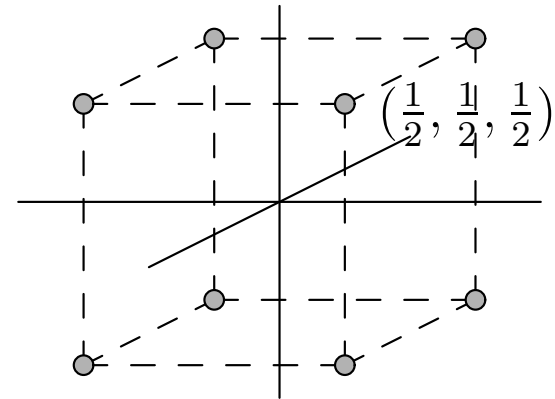
$USp(4)$



$SO(5)$



$SO(6)$



$USp(6)$

Grey points have positive \mathbb{Z}_2 -charge, while the white ones have negative.

The moduli space of vortices

The moduli space of vortices is completely described by the **moduli matrix**

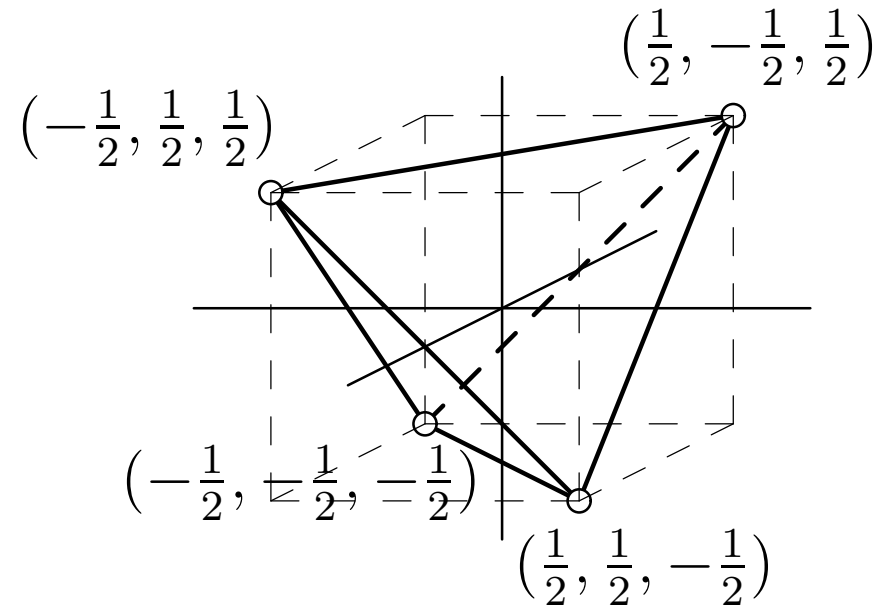
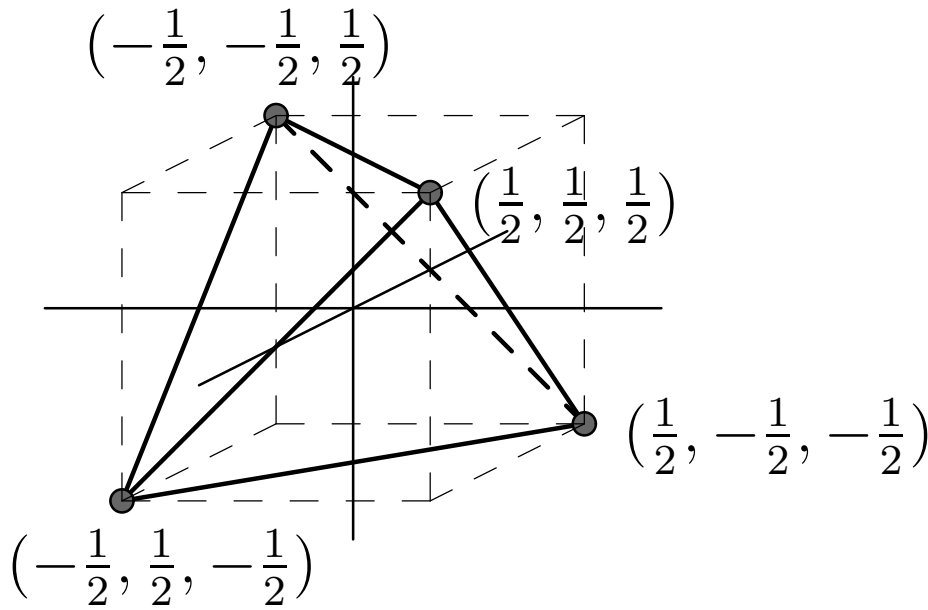
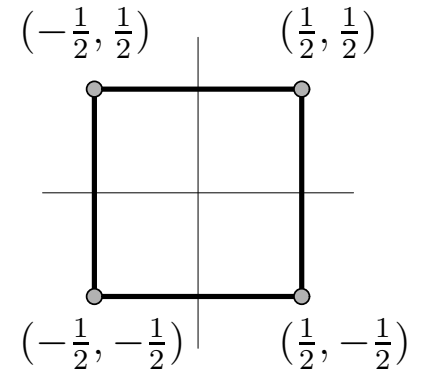
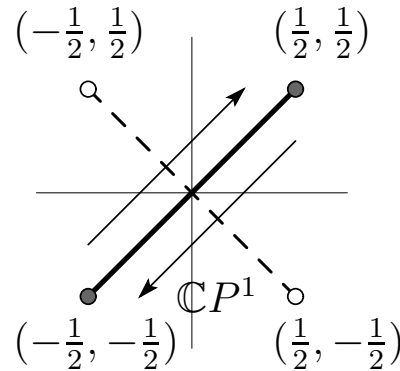
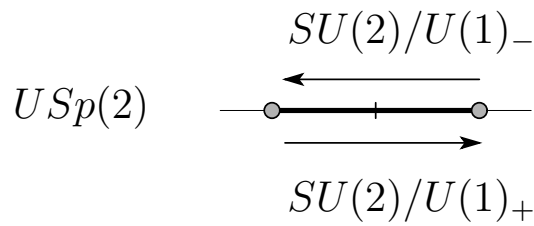
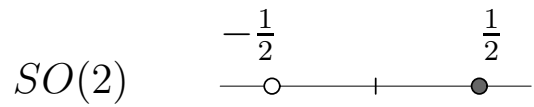
$$\mathcal{M} = \{H_0(z)\} // U(N)^{\mathbb{C}} . \quad (45)$$

Some examples: $k = 1$

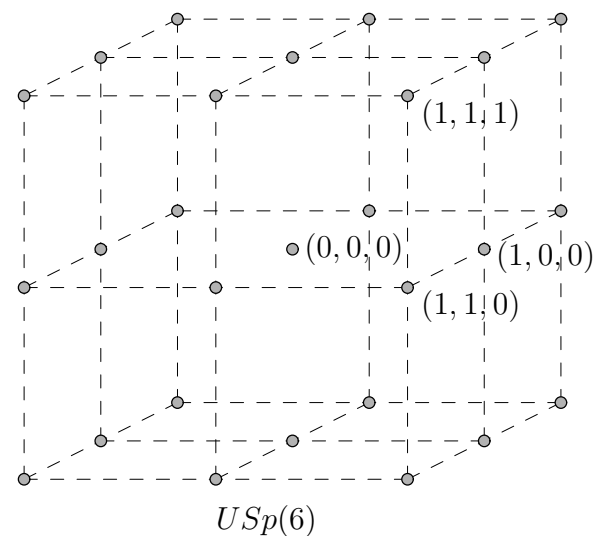
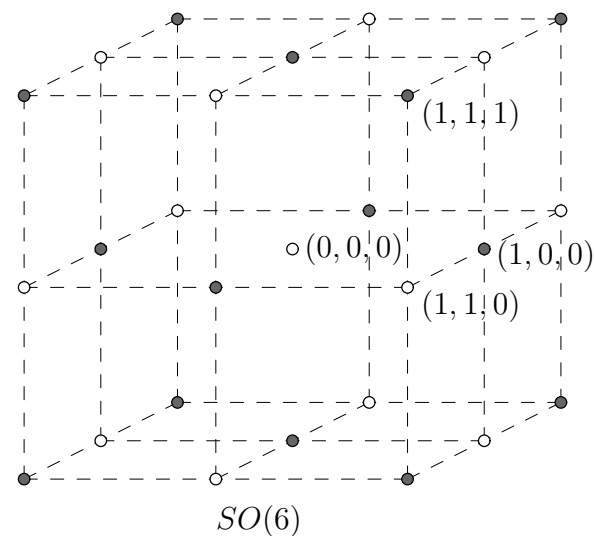
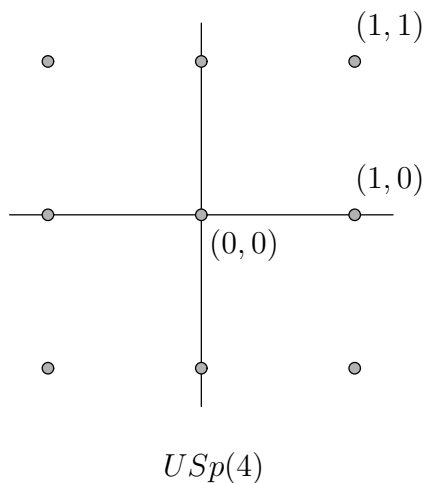
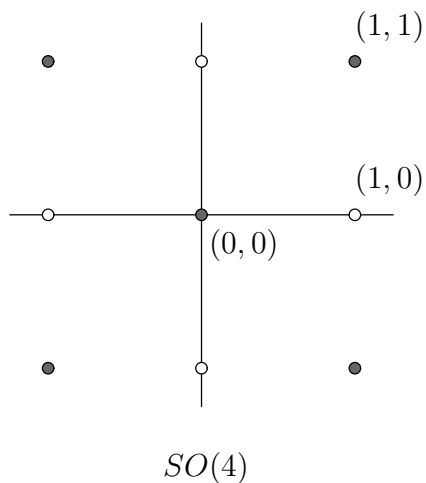
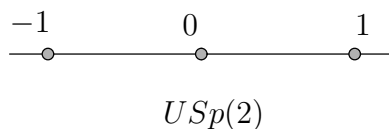
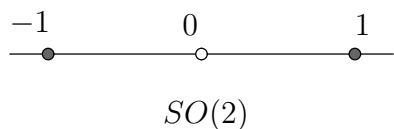
$$\mathcal{M}_{USp(2M)} = \mathbb{C} \times \frac{USp(2M)}{U(M)} , \quad (46)$$

$$\mathcal{M}_{SO(2M)} = \left(\mathbb{C} \times \frac{SO(2M)}{U(M)} \right)_+ \cup \left(\mathbb{C} \times \frac{SO(2M)}{U(M)} \right)_- , \quad (47)$$

Connectedness of the moduli space



The $k = 1$ odd SO cases resemble much the structure of the even cases with $k = 2$.



The master equations

By using the Ansatz for the first BPS equation $\bar{\mathcal{D}}H = 0$, we can write the other equations in terms of $\Omega = \omega\Omega' = SS^\dagger$ and $\Omega_0 = H_0(z)H_0^\dagger(z)$.

For $SU(N)$

$$\bar{\partial}\partial \log \omega = -\frac{e^2}{4N} \left[\frac{1}{\omega} \text{Tr} \Omega_0 \Omega'^{-1} - v^2 \right], \quad (48)$$

$$\bar{\partial} \left(\Omega' \partial \Omega'^{-1} \right) = \frac{g^2}{4\omega} \left[\Omega_0 \Omega'^{-1} - \frac{\mathbf{1}_N}{N} \text{Tr} \Omega_0 \Omega'^{-1} \right], \quad (49)$$

while for SO, USp

$$\bar{\partial} \left(\Omega' \partial \Omega'^{-1} \right) = \frac{g^2}{8\omega} \left[\Omega_0 \Omega'^{-1} - J^\dagger \left(\Omega_0 \Omega'^{-1} \right)^T J \right]. \quad (50)$$

So far we have not used the master equations **directly**, thus we have assumed the existence and uniqueness of the solutions to these equations. There are however two facts to support our assumption

1. Result of our index calculation with generic gauge group

number of bosonic zero-modes \Leftrightarrow number of moduli in $H_0(z)$

2. Strong coupling limit \Rightarrow non-linear σ -model **analytic solution is obtained**

Taking the strong coupling limit

For $SU(N)$

$$0 = \frac{1}{\omega} \text{Tr} \Omega_0 \Omega'^{-1} - v^2 , \quad (51)$$

$$0 = \Omega_0 \Omega'^{-1} - \frac{\mathbf{1}_N}{N} \text{Tr} \Omega_0 \Omega'^{-1} , \quad (52)$$

while for SO, USp

$$0 = \Omega_0 \Omega'^{-1} - J^\dagger \left(\Omega_0 \Omega'^{-1} \right)^T J , \quad (53)$$

which gives the solution for $SU(N)$

$$\omega_\infty = \frac{N}{v^2} (\det \Omega_0)^{\frac{1}{N}} , \quad \Omega'_\infty = (\det \Omega_0)^{-\frac{1}{N}} \Omega_0 , \quad (54)$$

while for SO, USp (where $M = H_0^T(z) J H_0$)

$$\omega_\infty = \frac{1}{v^2} \text{Tr} \sqrt{M^\dagger M} , \quad \Omega'_\infty = H_0(z) \frac{\mathbf{1}_N}{\sqrt{M^\dagger M}} H_0^\dagger(z) . \quad (55)$$

Lumps

Low energy
NL σ M
stringy lumps
 $H_0(z)$
target space



High energy
YM-Higgs
semi-local vortices
 $H_0(z)$
Higgs branch

The NL σ M is *integrable*.

Except for the *local vortex* which is mapped to a point.

Kähler quotient

For $U(N)$

$$\mathcal{L} = \text{Tr} \int d^4\theta \{ QQ^\dagger e^{-V} + \xi V \} , \quad (56)$$

$$\sim \int d^4\theta \xi \log \det QQ^\dagger , \quad (57)$$

For SO, USp

$$\mathcal{L} = \text{Tr} \int d^4\theta \{ QQ^\dagger e^{-V'} e^{-V_e} + \xi V_e \} , \quad (58)$$

where $e^{-V'} \in SO, USp$. Difficult calculation.

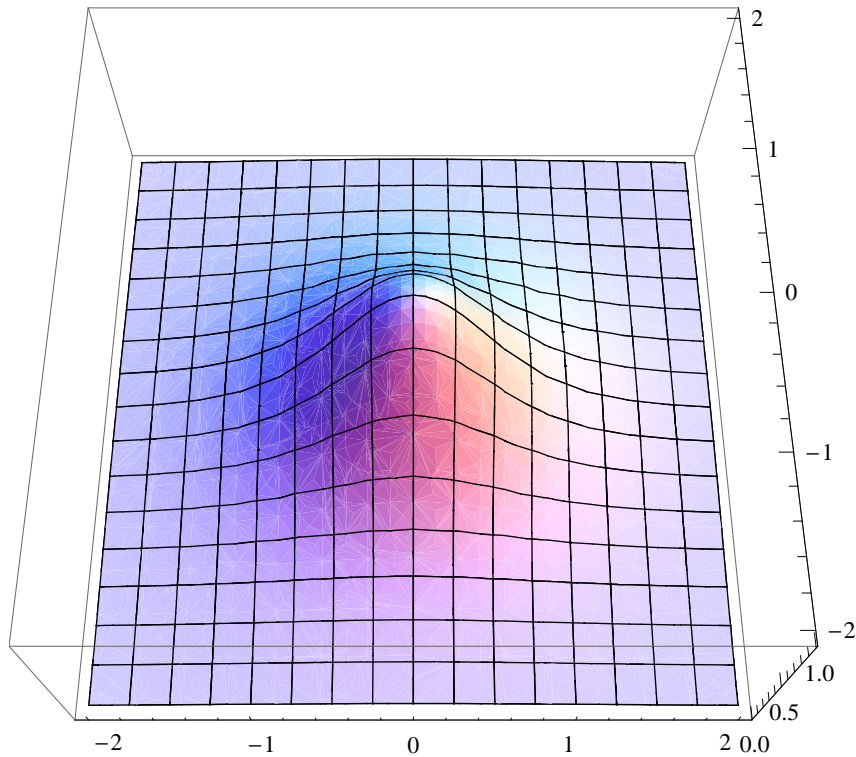
Kähler quotient for SO, USp

Relax the algebra $e^{-V'} \in SL(N, \mathbb{C})$ and introduce Lagrange multipliers λ .

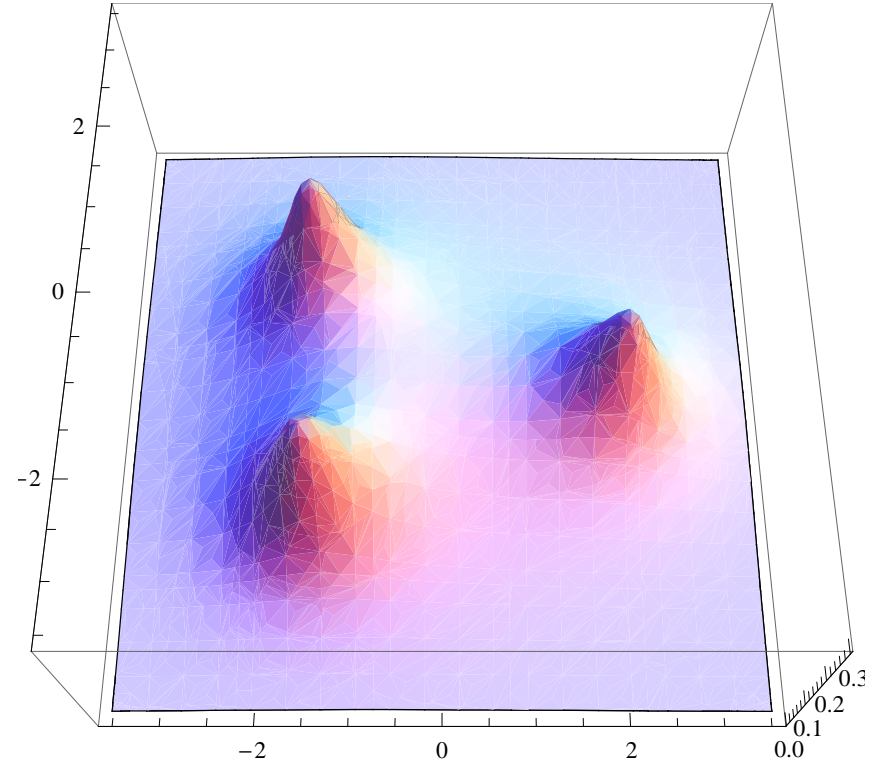
$$\begin{aligned} \mathcal{L} &= \text{Tr} \int d^4\theta \left\{ QQ^\dagger e^{-V'} e^{-V_e} + \lambda \left(e^{-V'^T} J e^{-V'} - J \right) + \xi V_e \right\} , \\ &\sim \int d^4\theta \xi \log \text{Tr} \sqrt{MM^\dagger} , \end{aligned} \quad (59)$$

where $M = Q^T J Q$ is the meson field. A similar construction has been made for the hyper-Kähler case ($\mathcal{N} = 2$).

Fractional vortices (lumps)



1



$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

Future developments

- Matching of monopole-vortex in SO, USp gauge theory.
- 2dim-4dim duality for vortices in SO, USp . $U(N)$ case solved by Hanany-Tong, Shifman-Yung.
- Composite solitons. $\frac{1}{4}, \frac{1}{8}$ BPS states.
- Q -balls in SO, USp .
- Exceptional groups, center-less groups.
- etc.

Thanks for your attention