
The pressure in the finite-temperature $(2 + 1)$ - d \mathbb{Z}_2 gauge theory

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The pressure as a function of the temperature is computed in the finite-temperature $(2 + 1)$ -dimensional \mathbb{Z}_2 lattice pure gauge theory, by means of a clever numeric approach which strongly reduces the computational effort. The proposed approach (first reported in by P. Kauf's thesis) makes use of locally deformed lattices to map the pressure into local observables with very small finite-size deviations and low statistical uncertainty. The measured quantities yield clear evidence for several glueball states in the confined phase, and well agree with the Stefan-Boltzmann prediction for high temperatures.

Outline of the talk

- ◇ Introduction
- ◇ The proposed method
- ◇ Extracting the physical pressure
- ◇ Results and conclusions
- ◇ References

Pressure in pure gauge lattice theories

Stefan-Boltzmann law: at high temperatures, a free bosonic gas in $(d + 1)$ dimensions has pressure

$$P_{\text{phys}}(T) = C_{SB} T^{d+1} ,$$

with a quantitative prediction for $C_{SB}(d)$. This is expected to hold for **any gauge theory with asymptotic freedom**.

The actual observation of this feature is very difficult: in facts, typical measurements report a value of C_{SB} which is lower than predicted (“**pressure deficit**”) even for temperatures several times the critical one.

This talk will present and explore a new approach to compute this observable [Kauf et al. '06]; although here it is presented for the \mathbb{Z}_2 case (where the **free-gas limit** for $T = \infty$ is **not guaranteed**), the mechanism is completely **generalisable**.

The data reported here are to be intended as somewhat preliminary (i.e. simulations are still running).

Standard approach

From the standard definition,

$$P(T) = \frac{T}{V} \log(Z) = \frac{1}{(aN_s)^d (aN_t)} \log(Z) \quad ,$$

a recipe is given in terms of the plaquette observable appearing in the Hamiltonian of the model [e.g. Bringoltz, Teper '05]:

$$a^{d+1} P_\beta(T) = \left(\frac{(d+1)!}{2} \right) \int_{\beta_0}^{\beta} d\beta' \left(\langle \square \rangle_{T;\beta'} - \langle \square \rangle_{0;\beta'} \right) \quad .$$

The **integration over couplings** is a major source of uncertainties; besides, to fix T the nonperturbative β -function would be needed.

Numerical evidence (for example, from $SU(3)$) suggests that the **Stefan-Boltzmann limit is reached very slowly** in the high temperature limit, making it difficult to connect with the idea of a free gas limit for QCD.

Our method

In the following, the model used will be the $(2 + 1)$ -dimensional \mathbb{Z}_2 gauge theory:

- enables nonlocal cluster algorithms for updating \Rightarrow faster simulations;
- lessons learned on dimensionally reduced, simplified theories usually carry over to more realistic theories.
- for this theory, glueball masses, critical indices and critical couplings are well known;
- can get rather close to continuum (lattice spacing ~ 0.02 fm);
- quite large systems accessible (spatial sides up to ~ 5 fm).

However, the algorithm in principle holds for the generic gauge theory in $(d + 1)$ dimensions.

Integration over couplings is absent, leading to a clean signal and statistical independence of data points; moreover, finite size corrections are exponentially vanishing.

Some notes on \mathbb{Z}_2 theory

By means of the **Kramers-Wannier duality**, this model is mapped onto the **Ising model** with coupling $\beta_I = -\frac{1}{2} \log(\tanh(\beta_G))$ and partition function $Z = \sum_c \exp(\beta_I \sum_{(x,y)} \sigma_x \sigma_y)$.

All simulations are performed in the Ising dual formulation, where [Talapov, Blöte '96]:

$$\beta_{3D}^C \simeq 0.2216544(3) \quad ; \quad \nu_{3D} \simeq 0.625(1) \quad .$$

The \hat{z} direction is compactified with radius $aN_t = 1/T$ (. . . which is not β).

Various couplings were investigated, each placing the transition at some critical N_t^C :

label	β_I	$N_t^C = 1/(aT_c)$	number of temperatures (conf. + deconf.)
c	0.227500	6.6215	7 + 5
d	0.226000	7.9262	13 + 6
e	0.224000	11.8517	11 + 11
m14	0.223346	14.5	13 + 13
m16	0.223027	16.5	14 + 0
m18	0.222795	18.5	0 + 10
3D	0.2216544	∞	-

Computing the Pressure I

The pressure (apart from subtraction of a constant) is given by:

$$P(T) = \frac{1}{(aN_t)(aN_s)^d} \log(Z_{N_t}) \quad ; \quad T = \frac{1}{aN_t} \quad .$$

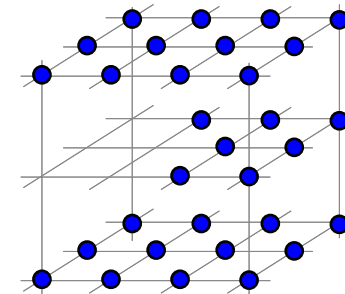
Now decompose the partition function in **ratios between two Z 's**: that of a system with $k - 1$ timeslices over that of a system with k :

$$Z_{N_t} = \frac{Z_1}{R_2 \cdot R_3 \cdots R_{N_t}} \quad ; \quad R_k \equiv \frac{Z_{k-1}}{Z_k} = \frac{Z \left(\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \right)}{Z \left(\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} \right)} \quad .$$

Computing the Pressure II

Now **apply again this decomposition** to the single ratio R_k : number the sites on a given timeslice with $i = 1, \dots, N_s^d$, and define:

$$R_k = \prod_i r_i^k, \quad r_i^k = \frac{Z_{(i)}^k}{Z_{(i-1)}^k},$$



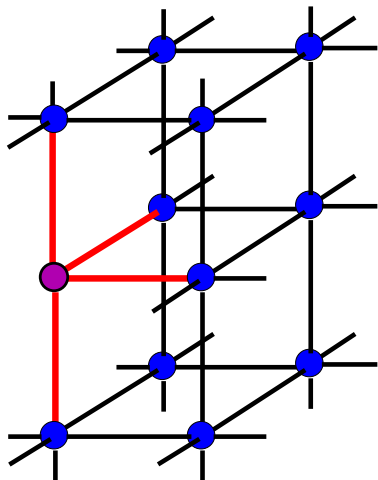
where the notation $Z_{(j)}^k$ means: the partition function for the system with k layers, but on one of them the first j sites have been **“removed”**.

If the $\{r_i^k\}$ turn out to have small variance and be quick to compute, **this setup is ready-to-go**. To achieve this, a good definition of “removed” is called for, along with a proper sequence of on-slice site removal.

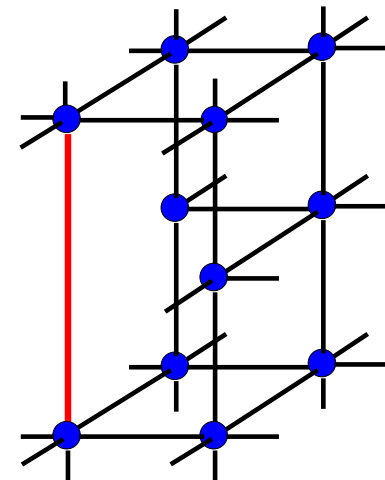
Computing the Pressure III : site-removal rules

Number the sites in vertical lines upward, starting from the leftmost in a systematic way. This will be the site-removal ordering.

5	10	↑	↑	↑	↑	↑	↑	↑	45
4	9								↑
3	8			23					
2	7			22					
1	6	11	21						



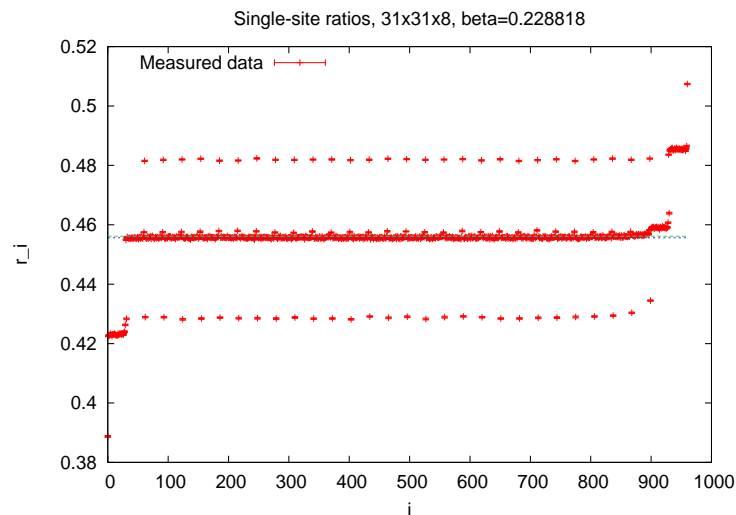
The “smoothest” way to make the timeslice disappear completely from $Z_{(j)}^k$ as j increases is the following: when a site is removed, its spatial links simply vanish, while in place of the two vertical links a new direct link connecting the two ex-second-neighbours pops up.



this sketch can be adapted to the cases of “removee” site on system border.

Numerical outcome of this recipe

The single-site removal ratios $\{r_i^k\}$ actually fluctuate mildly, so that the final timeslice ratio R_k will confer a reasonable precision to $P(T)$. But what really counts is that . . .



. . . few nodes are displaced from their logarithmic average, namely those strongly feeling the timeslice boundaries. But they are a minority for large spatial sides, so **considering only one single r_i not only will suffice, but also will give less finite-size deviations!** Moreover, its value of r_i has a **rapid convergence to the thermodynamic limit** of $(R_k)^{1/N_s^d}$.

In short: less simulations \Rightarrow better measurement !

Computing the Pressure IV : the single-site ratio

The key observable can be rewritten as:

$$r_i^k = \frac{1}{2} \left\langle e^{\beta I [H_{(i-1)}^k - H_{(i)}^k]} \right\rangle_{Z_{(i-1)}^k},$$

that is, the expectation value of the exponentiated **energy difference induced by removing site i** . Its value amounts to the product of $\sigma_x \sigma_y$ summed over added or removed links, with sign $+$ or $-$ respectively; apart from on-border cases:

$$H_{(i-1)}^k - H_{(i)}^k = \left[\left(\begin{array}{c} \bullet \\ | \\ \circ \\ | \\ \bullet \\ | \\ \bullet \end{array} \right) - \left(\begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \right) \right].$$

The average is to be computed with the **dynamics provided by the Z with the node still present** (although other choices are possible).

Note: from now on, only $r^k \equiv r_{\frac{1}{2}(N_s)d}^k$ will be considered (centered in timeslice).

5	10	↑	↑	↑	↑	↑	↑	45
4	9							↑
3	8			23				
2	7			22				
1	6	11		21				

Computing the Pressure V : extraction of $P(T)$

The physically relevant pressure is obtained after subtraction of the zero-temperature contribution:

$$\Delta P(T) \equiv P(T) - P_\infty \quad .$$

By rewriting $\log(Z)$ in terms of the ratios $\{r^k\}$, two approaches are possible:

- **Direct fit** of the measured ratios to the form:

$$\frac{1}{a^{d+1}} \log(r^k) = (k-1)\Delta P(T_{k-1}) - k\Delta P(T_k) - P_\infty \quad , \quad T_\ell = 1/(a\ell) \quad ,$$

with some physically motivated $\Delta P(T)$. All fits regard P_∞ as the only parameter to be fixed.

- **Resum the ratios** in order to reconstruct the full $\Delta P(T)$; since $\lim_{k \rightarrow \infty} \log(r^k) = -P_\infty$,

$$a^{d+1} \Delta P(T_k) = \frac{1}{k} \sum_{j=k+1}^{\infty} (\log(r^j) + P_\infty) \quad .$$

The latter approach is numerically very demanding and will not be pursued here.

Functional forms for the pressure: confined phase

◇ In the confined phase, the pure gauge model possesses **massive glueball states**. In this case the pressure integral (assuming free glueballs),

$$P(T) = \sum_i \int \frac{d^2 p}{(2\pi)^2} \left[-kT \log \left(1 - e^{-\frac{\omega_i(p)}{kT}} \right) \right] , \quad \omega^{(i)} = \sqrt{M_i^2 + p^2} ,$$

provides:

$$P(T) = \sum_i \left[\frac{T^3}{2\pi} \sum_{k=1}^{\infty} \frac{1}{k^3} \left(1 + k \frac{M_i}{T} \right) e^{-k \frac{M_i}{T}} \right] + P_{\infty} , \quad T < T_c ,$$

which is usually truncated to the first lightest mass(es) out of an infinite tower of states.

Notice that in $(2 + 1)$ dimensions the spin- J **multiplicity factor $(2J + 1)$ is absent**.

Functional forms for the pressure: deconfined phase

◇ In the deconfined phase, the system is a **gas of massless bosons**; dimensional analysis implies, in $(d + 1)$ dimensions:

$$P(T) = CT^{d+1} + P_\infty \quad , \quad T \gg T_c \quad .$$

In case of *noninteracting* gas (Stefan-Boltzmann), the prediction is quantitative,

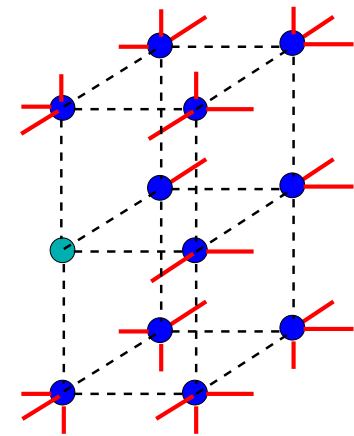
$$C_{SB} = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \zeta(d+1) \quad ,$$

and in particular $C_{SB} = \frac{\zeta(3)}{2\pi} \simeq 0.191313$ for the $(2 + 1)$ -dimensional free gas.

If the particles (**gluons**) have **multiplicity** μ , we simply have μC in place of C .

Simulation facts

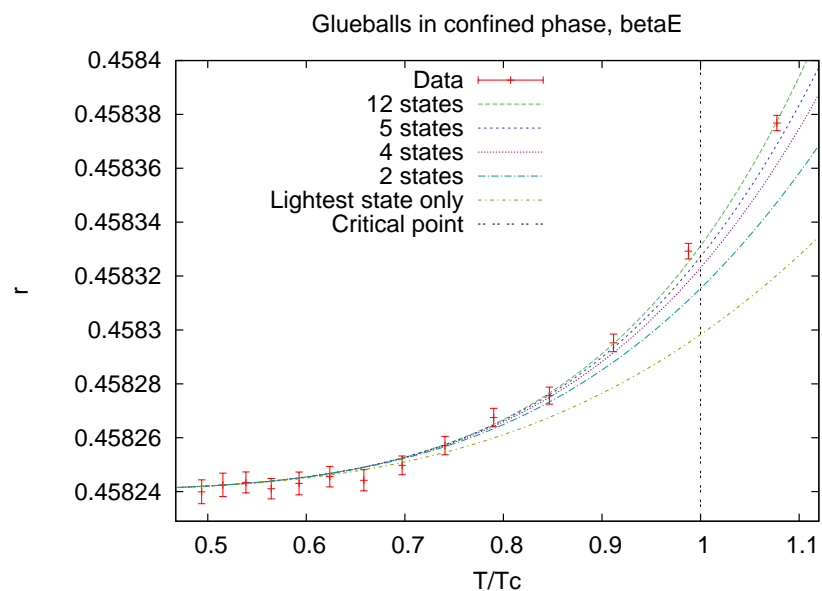
- Nonlocal cluster update with the Swendsen-Wang algorithm.
- Exact integration is implemented over the 12 sites surrounding the “remove” node: this implies measuring the state of the 29-site shell surrounding the integrated region. This allows for a great variance reduction.
- The confined phase shows no signs of finite-size effects; in the high-temperature regime the convergence is exponential: $r(N_s) = r_\infty + a \exp(-(\frac{N_s}{\mathcal{N}_0})^2)$, with $\mathcal{N}_0 \sim \mathcal{O}(20)$.
- Efficiency: $2.2 \cdot 10^{-7}$ seconds per update per lattice site.



Using spatial sides $N_s \sim 60$ to 80 lattice spacings, our statistics were of the order $10^6 - 10^7$ measurements for each β and each T .

Results I: confined phase

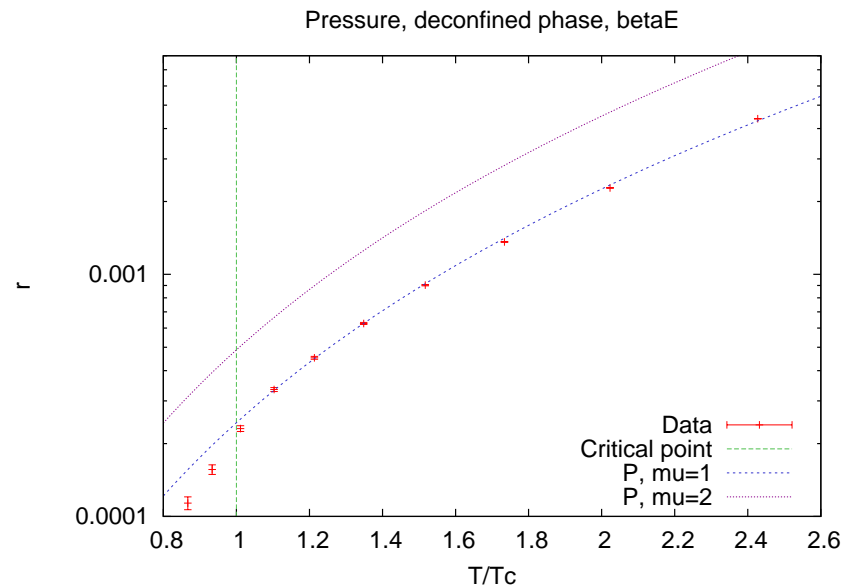
At temperatures lower than T_c , the measured ratios $\{r^k\}$ are well described by the massive glueball gas formula for $P(T)$: the agreement increases by adding heavier and heavier glueballs.



The final curve includes *twelve* glueballs from all spin-parity channels and describes data up to nearly T_c (in agreement with usual hadron resonance gas ideas).

Results II: deconfined phase

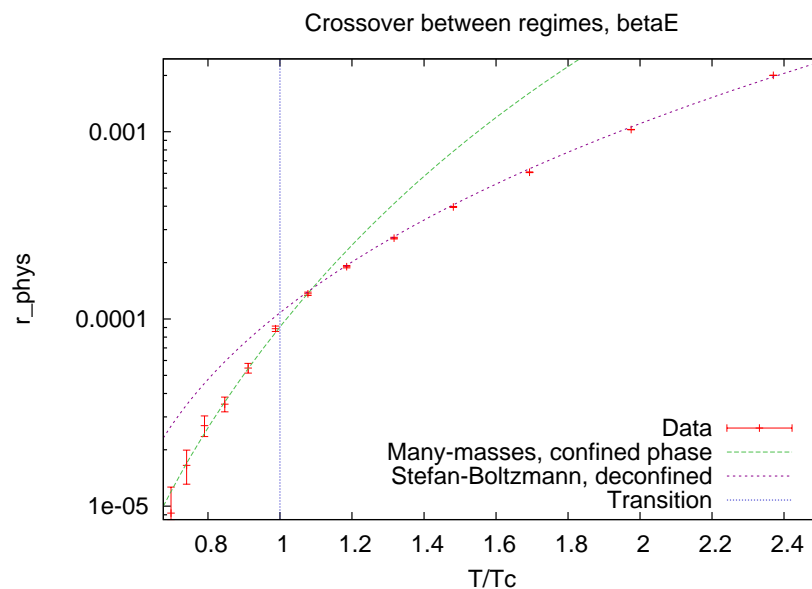
The free bosonic S–B formula well matches the data starting from temperatures $\sim 1.1 T_c$:



Compare to the curve for the case of multiplicity-2 particles: this model clearly possesses a **single “gluon”**.

Results III: overall behaviour of $P(T)$

A nice **crossover** between the confined and the deconfined regime is observed in a narrow region around criticality:

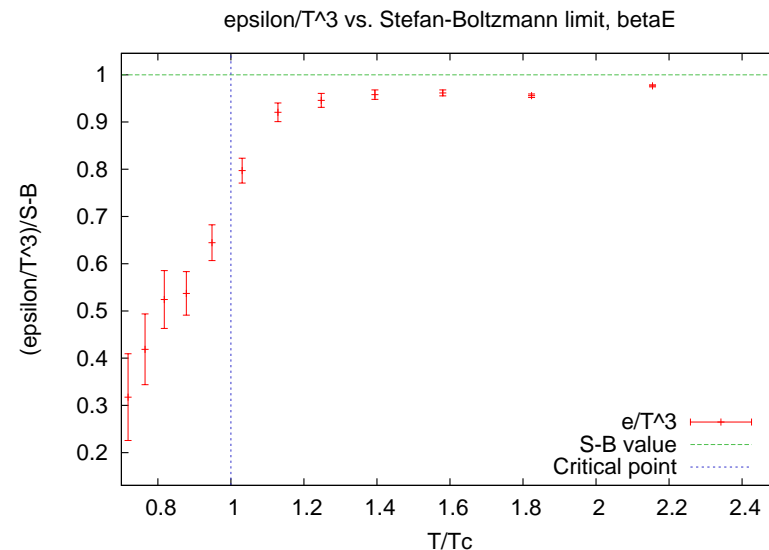


The S–B behaviour cannot be observed at very high temperatures due to the onset of large discretisation errors. Finer lattices are expected to allow going to larger T/T_c .

Results IV: internal energy

$U = d(\beta F)/d\beta \Rightarrow$ the ratios $\{r^k\}$ encode the density of internal energy ϵ ; up to $\mathcal{O}(a^2)$,

$$\epsilon(k - 1/2) = kP_k - (k - 1)P_{k-1} = -P_\infty - \log(r^k) \quad ;$$



The limit $\frac{\zeta(3)}{\pi}$ is almost fulfilled very early: \mathbb{Z}_2 theory is practically free just after deconfinement (although the small residual deficit shown here deserves further investigation).

Summary & outlook

- The algorithm described here allows to skip many of the problematic issues in usual approaches to the pressure on the lattice; it can be virtually applied to any lattice theory and provides precise results in a reasonable time.
- The method works at fixed lattice spacing and no knowledge of the beta-function is needed [see Hatsuda et al. '08].
- fast approach to Stefan-Boltzmann for $T > T_c$; precise validation of hadron resonance gas for $T < T_c$ [see Meyer '09], with no signs of glueball interactions whatsoever.
- This work focused on the \mathbb{Z}_2 theory, showing that its pressure is in good agreement with the theoretical prediction both above and below the transition, with a sharp crossover between the two regimes.
- The next step will be to investigate on $SU(2)$ theory along the same lines: some technical difficulties will need careful consideration to keep the algorithm performance as high as in this case.

References

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Different couplings - deconfined phase (preliminary)

