

CNISM, Unità di Ricerca di Salerno, Dipartimento di Fisica "E. R. Caianiello", Università di Salerno and Dipartimento di Scienze Fisiche, Università di Napoli "Federico II"

A twisted CFT approach to the study of antiferromagnetic two-leg spin-1/2 ladders

with Mobius boundary conditions

Adele Naddeo

In collaboration with:
Gerardo Cristofano (Napoli),
Vincenzo Marotta (Napoli),
Giuliano Niccoli (DESY, Amburgo)

G. Cristofano, V. Marotta, A. Naddeo, G. Niccoli, JSTAT (2008) P12010.

Outline

- Introduction: 2-leg XXZ spin-1/2 ladders with Mobius boundary conditions
- The XXZ spin-1/2 chain in the continuum limit
- 2-reduction technique on the plane: generalities, the 2-leg XXZ spin-1/2 ladder, the special XXX case
- Renormalization group analysis for the XXX case

Depending on M , we get two topologically inequivalent boundary conditions:

1. M even: we get two independent XXZ spin-1/2 chains, each one closed by gluing opposite ends, that is periodic boundary conditions (PBC).
2. M odd: the two legs are not independent, they appear to be connected at a point upon gluing the opposite ends and the system can be viewed as a single eight shaped chain, that is Mobius boundary conditions (MBC).



For **M odd** the system presents a local topological defect in the gluing point: it coincides with two closed XXZ spin-1/2 chains, each one with $(M+1)/2$ sites, which intersect each other in the topological defect at site $M+1 \equiv 1$. The general Hamiltonian in this case is written as:

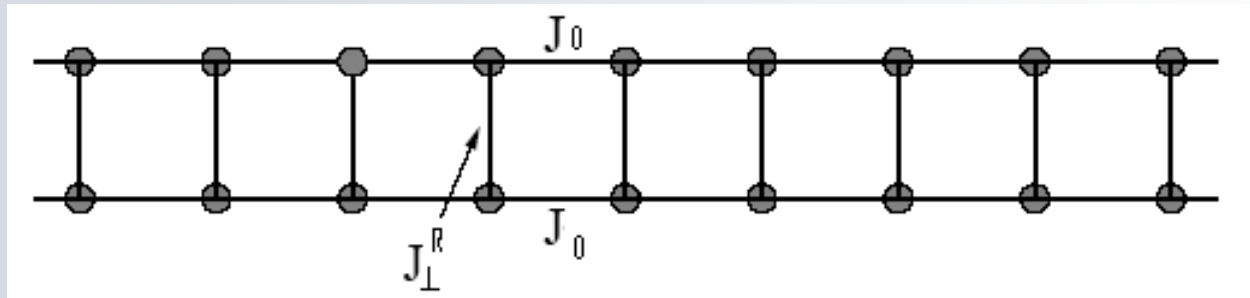
$$H_0^{MBC} = H_0 + H_{MBC}$$
$$H_{MBC} = J_0 \sum_{i=M, M+1} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) - J_0 \left(S_M^x S_{M+2}^x + S_M^y S_{M+2}^y + \Delta S_M^z S_{M+2}^z \right)$$

2. Weak interacting system: we introduce the following interactions between the two spin-1/2 XXZ chains: railroad, zigzag, 4-spin, 4-dimer.

Railroad

$$H_0 + H_{\text{Railroad}}$$

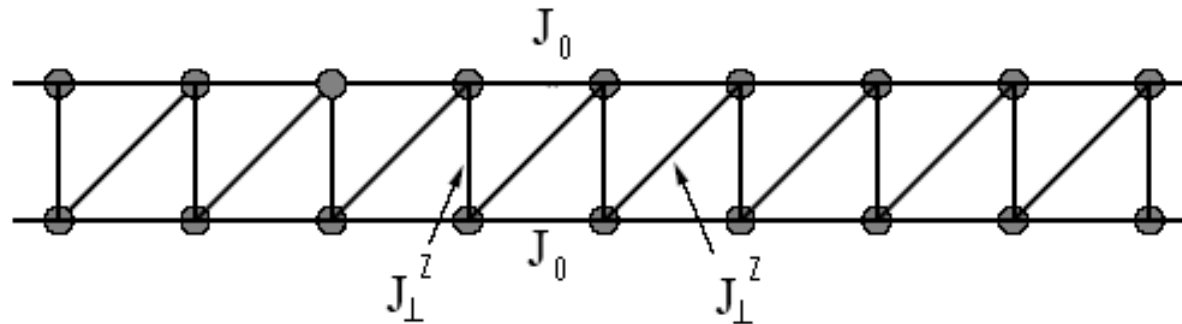
$$H_{\text{Railroad}} = J_{\perp}^R \sum_{i=1}^{[(M+1)/2]} \left(S_{2i-1}^x S_{2i}^x + S_{2i-1}^y S_{2i}^y + \Delta S_{2i-1}^z S_{2i}^z \right)$$



In the weak coupling limit $J_0 \gg J_{\perp}^R$, for the isotropic XXX case ($\Delta=1$), we have a gapful spectrum with the formation of massive spin $S=1$ and $S=0$ particles (D. G. Shelton, A. A. Nersesian, A. M. Tsvelick, PRB 53 (1996) 8521).

Zigzag

$$H_0 + H_{\text{Zigzag}}$$
$$H_{\text{Zigzag}} = J_{\perp}^Z \sum_{i=1}^M \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right)$$



In the weak coupling limit $J_0 \gg J_{\perp}^Z$, for the isotropic XXX case ($\Delta=1$), an exponentially small gap develops and the weak interchain correlations break translational symmetry. There is a spontaneous dimerization along with a finite range incommensurate magnetic order (A. A. Nersesyan, A. O. Gogolin, F. H. L. Essler, PRL 81 (1998) 910; D. Allen, D. Senechal, PRB 55 (1997) 299).

4-Spin

$$H_{4\text{-Spin}} = J_{\varepsilon} \sum_{i=1}^{[(M+1)/2]} \varepsilon_i^{Dw} \varepsilon_i^{Up}$$

$$\varepsilon_i^{Dw} = (-1)^i \left(S_{2i-1}^x S_{2i+1}^x + S_{2i-1}^y S_{2i+1}^y + \Delta S_{2i-1}^z S_{2i+1}^z \right)$$

$$\varepsilon_i^{Up} = (-1)^i \left(S_{2i}^x S_{2i+2}^x + S_{2i}^y S_{2i+2}^y + \Delta S_{2i}^z S_{2i+2}^z \right)$$

Such a coupling can be generated either by phonons or by the conventional Coulomb repulsion between the holes.

4-Dimer

$$H_{4\text{-Dimer}} = J_{\kappa} \sum_{i=1}^{[(M+1)/2]} \kappa_i \kappa_{i+1}$$

$$\kappa_i = \varepsilon_i^{Dw} \varepsilon_i^{Up}$$


Such a term plays a crucial role in the presence of the marginal interaction H_{ZigZag} only, because it gives rise to the dynamical generation of the triplet or singlet mass respectively along two different *RG* flows.

The XXZ spin-1/2 chain in the continuum limit (abelian bosonization)

$$H_{XXZ} = J \sum_{j=1}^N \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right)$$

Jordan-Wigner transformation:

$$S_j^z = n_j - \frac{1}{2}, \quad S_j^- = (-1)^j c_j e^{-i\pi \sum_{l=1}^{j-1} n_l}, \quad S_j^+ = (-1)^j c_j e^{i\pi \sum_{l=1}^{j-1} n_l}$$



$$H_{XXZ} = J \sum_{j=1}^N \left[-\frac{1}{2} (c_{j+1}^+ c_j + c_j^+ c_{j+1}) + \Delta \left(n_j - \frac{1}{2} \right) \left(n_{j+1} - \frac{1}{2} \right) \right]$$

The Fermi surface is made of two points, $k_F = \pm\pi/2$, and linearizing the energy spectrum around them we get in the continuum:

$$c_j \rightarrow \psi(x) = e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x)$$

where $x=aj$, for lattice spacing a , and $\psi_R(x)$ and $\psi_L(x)$ correspond to right and left moving fermions, which can be bosonized according to:

$$\psi_{R,L}(x) \approx e^{-i\phi_{R,L}(x)}$$

By taking the continuum limit and keeping only the marginal operators, we get the exactly solvable Luttinger Hamiltonian:

$$H_{LL} = \frac{v}{8\pi} \int dx \left[(\partial_x \theta)^2 + (\partial_x \phi)^2 \right]$$

where we defined:

$$\phi = \frac{\phi_L + \phi_R}{\sqrt{K}}, \quad \theta = \sqrt{K}(\phi_L - \phi_R)$$

$$g = \frac{(2J\Delta \sin^2 k_F)}{\pi v_F}, \quad v = \frac{v_F}{2} \sqrt{1+2g}, \quad K = \frac{1}{\sqrt{1+2g}}$$

renormalized velocity

Luttinger parameter

A comparison with exact Bethe Ansatz calculations leads to the expressions:

$$v(\Delta) = \frac{J\pi}{2} \frac{\sqrt{1-\Delta^2}}{\arccos \Delta}, \quad K(\Delta) = \frac{\pi}{2(\pi - \arccos \Delta)}$$

The assumption of periodic boundary conditions implies the compactification of the boson fields:

$$\begin{aligned} \phi(x+L, t) &= \phi(x, t) + 2\pi m_\phi R_\phi, & m_\phi &\in \mathbb{Z}, R_\phi = \frac{1}{\sqrt{K}} \\ \theta(x+L, t) &= \theta(x, t) + 2\pi m_\theta R_\theta, & m_\theta &\in \mathbb{Z}, R_\theta = 2\sqrt{K} \end{aligned}$$

The compactified boson field has the following mode expansion:

$$\begin{aligned} \phi(z, \bar{z}) &= q - ia_0 \ln z - i\bar{a}_0 \ln \bar{z} + \sum_{n \neq 0} \frac{ia_n}{n} z^{-n} + \sum_{n \neq 0} \frac{i\bar{a}_n}{n} \bar{z}^{-n} \\ z &= e^{2\pi(\tau-ix)/L}, \quad \bar{z} = e^{2\pi(\tau+ix)/L}, \quad \tau = ivt \\ a_0 &= p + w, \quad \bar{a}_0 = p - w \\ [q, p] &= i, \quad [a_n, a_{n'}] = n\delta_{n,n'}, \quad [a_n, \bar{a}_{n'}] = 0, \quad [\bar{a}_n, \bar{a}_{n'}] = n\delta_{n,n'} \end{aligned}$$

2-reduction technique on the plane: generalities

(G.Cristofano, G. Maiella, V.Marotta, MPLA **15**, 1679 (2000); G. Cristofano, G. Maiella, V. Marotta, G. Niccoli, NPB **641**, 547 (2002); G. Cristofano, V. Marotta, A. Naddeo, PLB **571**, 250 (2003); G. Cristofano, V. Marotta, A. Naddeo, NPB **679**, 621 (2004))

Mother theory: a CFT with $c=1$, described in terms of a boson field $\phi(z, \bar{z})$ compactified on a circle with general radius R_ϕ . We can define two scalar fields, symmetric and antisymmetric under Z_2 :

$$\tilde{X}(\omega, \bar{\omega}) = \frac{\phi(\omega, \bar{\omega}) + \phi(e^{i\pi}\omega, e^{-i\pi}\bar{\omega})}{\sqrt{2}}, \quad \tilde{\Phi}(\omega, \bar{\omega}) = \frac{\phi(\omega, \bar{\omega}) - \phi(e^{i\pi}\omega, e^{-i\pi}\bar{\omega})}{\sqrt{2}}$$

Daughter theory: we implement the map $z \rightarrow \omega^2$, getting an orbifold CFT with $c=2$. The new theory is described in terms of a Z_2 -invariant scalar field X and a twisted field ϕ which satisfies **twisted boundary conditions:**

$$X(z, \bar{z}) = \tilde{X}(z^{1/2}, \bar{z}^{1/2}), \quad \Phi(z, \bar{z}) = \tilde{\Phi}(z^{1/2}, \bar{z}^{1/2})$$

The mode expansion of these two fields is:

$$\begin{aligned}
 X(z, \bar{z}) &= q_0 - i\alpha_0 \ln z - i\bar{\alpha}_0 \ln \bar{z} + \sum_{n \in \mathbb{Z} - \{0\}} \frac{i\alpha_n}{n} z^{-n} + \sum_{n \in \mathbb{Z} - \{0\}} \frac{i\bar{\alpha}_n}{n} \bar{z}^{-n} \\
 \Phi(z, \bar{z}) &= -\sqrt{2}\pi w + \sum_{n \in \mathbb{Z}} \frac{i\alpha_{n+1/2}}{n+1/2} z^{-(n+1/2)} + \sum_{n \in \mathbb{Z}} \frac{i\bar{\alpha}_{n+1/2}}{n+1/2} \bar{z}^{-(n+1/2)} \\
 q_0 &= \sqrt{2}(q + \pi w), \quad \alpha_{n+1/2} = \frac{a_{2n+1}}{\sqrt{2}}, \quad \bar{\alpha}_{n+1/2} = \frac{\bar{a}_{2n+1}}{\sqrt{2}}, \quad n \in \mathbb{N}; l = 0, 1
 \end{aligned}$$

where:

$$\begin{aligned}
 [\alpha_{n+1/2}, \alpha_{n'+1'/2}] &= \left(n + \frac{l}{2}\right) \delta_{n,n'} \delta_{l,l'}, & [\bar{\alpha}_{n+1/2}, \bar{\alpha}_{n'+1'/2}] &= \left(n + \frac{l}{2}\right) \delta_{n,n'} \delta_{l,l'} \\
 [\alpha_{n+1/2}, \bar{\alpha}_{n'+1'/2}] &= 0, & [q_0, \alpha_{n+1/2}] &= i\delta_{0,n} \delta_{0,l}, & [q_0, \bar{\alpha}_{n+1/2}] &= i\delta_{0,n} \delta_{0,l}
 \end{aligned}$$

The field X is a compactified boson with compactification radius $R_X = R_\phi / \sqrt{2}$.

2-reduction technique on the plane: the 2-leg XXZ spin-1/2 ladder


We show that the Twisted Model (TM), generated by 2-reduction technique, describes the continuum limit of the 2-leg XXZ spin-1/2 ladder with PBC and MBC. It is enough to show that the XXZ ladder with M odd and MBC is naturally mapped in the twisted sector of the TM. Such a system coincides with a system of 2 XXZ chains, each one with $(M+1)/2$ sites and size $L=a(M+1)/2$, which are closed with periodicity condition in one gluing site common to the 2 chains.



Upon bosonizing each chain, we obtain two boson fields ϕ_{Up} and ϕ_{Dw} compactified on the two circles (up and down) of the same length L and with the same compactification radius $R_\phi = \sqrt{2}(\pi - \arccos\Delta)/\pi = 1/\sqrt{K}$.

The topological defect implies the following b.c. for the fields at the gluing point:

$$\phi_{Up}(0, t) = \phi_{Dw}(0, t)$$


$$\phi(x, t) = \begin{cases} \phi_{Dw}(x, t), & 0 \leq x \leq L \\ \phi_{Up}(x - L, t), & L \leq x \leq 2L \end{cases}$$

Folding field

The compactification space of the field ϕ is an eight of length $2L$ and the radius is R_ϕ .

$$\phi(x + 2L, t) = \begin{cases} \phi_{Dw}(x + L, t), & 0 \leq x \leq L \\ \phi_{Up}(x, t), & L \leq x \leq 2L \end{cases}$$

$$= \begin{cases} \phi_{Dw}(x, t) + 2\pi m_\phi R_\phi, & 0 \leq x \leq L \\ \phi_{Up}(x - L, t) + 2\pi m_\phi R_\phi, & L \leq x \leq 2L \end{cases}$$



$$\phi(x + 2L, t) = \phi(x, t) + 2\pi m_\phi R_\phi$$

2-reduction

Basic steps

$$\omega = e^{2\pi(\tau - ix)/2L}, \quad \bar{\omega} = e^{2\pi(\tau + ix)/2L}, \quad \tau = ivt$$

$$\tilde{X}(\omega, \bar{\omega}), \quad \tilde{\Phi}(\omega, \bar{\omega})$$

$$z = \omega^2 = e^{2\pi(\tau - ix)/L}, \quad \bar{z} = \bar{\omega}^2 = e^{2\pi(\tau + ix)/L}$$

$$X(z, \bar{z}), \quad \Phi(z, \bar{z})$$

By means of 2-reduction technique we have transformed the boson field ϕ compactified on the circle of length $2L$ with radius R_ϕ into the two independent boson fields X and Φ compactified on a circle of length L , where $R_X = R_\phi/\sqrt{2}$. Thus the 2-leg XXZ spin-1/2 ladder with general anisotropy Δ and MBC is described by the twisted sector of the TM.

The special XXX case

Let us put $\Delta=1$, which corresponds to the isotropic XXX ladder.

Mother theory: a CFT with $c=1$, described in terms of a boson field compactified on a circle with radius $R_\phi = \sqrt{2}$ ($K=1/2$).

Daughter theory: as a result of the 2-reduction procedure, we get an orbifold CFT with $c=2$. The new theory is described in terms of two boson fields, X and ϕ , which describe the spin chains of the two legs. It decomposes into a tensor product of two CFTs, a twisted invariant one with $c=3/2$, realized by the boson X and a Ramond Majorana fermion, while the second has $c=1/2$ and is realized in terms of a Neveu-Schwartz Majorana fermion: $su(2)_2 \otimes I$.

Renormalization group analysis for the XXX case

Let us consider the weakly interacting 2-leg ladder in the isotropic case, i. e. XXX, and study the different RG trajectories flowing from the UV fixed point described by our TM with central charge $c=2$.

The daughter fields X and Φ and their duals admit the following representation in terms of left and right chiral components:

$$\begin{aligned} X(z, \bar{z}) &= -i w_X \ln \frac{z}{\bar{z}} + X(z) + \bar{X}(\bar{z}), & Y(z, \bar{z}) &= X(z) - \bar{X}(\bar{z}) \\ \Phi(z, \bar{z}) &= -2\pi w_X + \varphi(z) + \bar{\varphi}(\bar{z}), & \Theta(z, \bar{z}) &= \varphi(z) - \bar{\varphi}(\bar{z}) \end{aligned}$$

A representation in terms of four Majorana fermion fields gives (for holomorphic and anti-holomorphic components):

$$\begin{aligned} \psi_1(z) &= \sin X(z), & \psi_2(z) &= \cos X(z), & \psi_3(z) &= \frac{\sin \varphi(z)}{\sqrt{z}}, & \psi_0(z) &= \frac{\cos \varphi(z)}{\sqrt{z}} \\ \bar{\psi}_1(\bar{z}) &= -\sin \bar{X}(\bar{z}), & \bar{\psi}_2(\bar{z}) &= \cos \bar{X}(\bar{z}), & \bar{\psi}_3(\bar{z}) &= -\frac{\sin \bar{\varphi}(\bar{z})}{\sqrt{\bar{z}}}, & \bar{\psi}_0(\bar{z}) &= \frac{\cos \bar{\varphi}(\bar{z})}{\sqrt{\bar{z}}} \end{aligned}$$

The Lagrangian describing the UV fixed point for the XXX ladder is:

$$L_0 = \frac{1}{8\pi} \left(\partial_\mu X \partial^\mu X + \partial_\mu \Phi \partial^\mu \Phi \right)$$

while the perturbing terms V depend on the particular system under study:

$$L = L_0 - V$$

1. Railroad and 4-Spin perturbations: massive flow

$$V_{\text{Railroad}} = -m_R \left(\cos X(z, \bar{z}) - \cos \Phi(z, \bar{z}) + 2 \cos \Theta(z, \bar{z}) \right), \quad m_R \propto J_\perp^R$$

$$V_{\text{Railroad}} = -im_R \sum_{i=1}^3 \psi_i(z) \bar{\psi}_i(\bar{z}) + 3im_R \psi_0(z) \bar{\psi}_0(\bar{z})$$

Majorana fermion representation

This is a relevant perturbation to the UV fixed point; the ψ_i ($i=1,2,3$) fields form an Ising triplet with the same mass m_R ($su(2)_2$ sector) and ψ_0 is an Ising singlet ($/$ sector) with a larger mass $-3m_R$ (D. G. Shelton, A. A. Nersesyan, A. M. Tsvelick, PRB 53 (1996) 8521).

$$V_{4-Spin} = m_\varepsilon (\cos X(z, \bar{z}) + \cos \Phi(z, \bar{z})), \quad m_\varepsilon \propto J_\varepsilon$$

$$V_{4-Spin} = im_\varepsilon \sum_{i=0}^3 \psi_i(z) \bar{\psi}_i(\bar{z})$$

A. A. Nersesyan, A. M. Tsvelick,
PRL 78 (1997) 3939

Such a relevant mass term gives rise to the same mass contribution m_ε for all the Ising fields ψ_i ($i=0,1,2,3$). That allows the triplet or singlet mass to vanish also for finite values of the coupling constants, i.e. far from the UV conformal fixed point $c=2$

We get two possible RG flows: a flow to an IR fixed point with $c=3/2$ as a result of the Ising / decoupling ($m_t=0, m_s \neq 0$); a flow to a different IR fixed point with $c=1/2$ as a result of the $su(2)_2$ decoupling ($m_t \neq 0, m_s = 0$).

2. Zigzag and 4-Dimer perturbations: massive flow

Let us write the non interacting Lagrangian in terms of Majorana fermions only and observe that the simple continuum limit, without adding the zigzag perturbation, produces a marginal interaction V_U (D. Allen, D. Senechal, PRB 55 (1997) 299):

$$L_0 = \frac{1}{2\pi} \sum_{i=0}^3 v_i (\psi_i \bar{\partial} \psi_i + \bar{\psi}_i \partial \bar{\psi}_i), \quad v_0 = \dots = v_3 = v \approx J_0 a$$

$$\begin{aligned}
V_U &= -\lambda_U (O_1 + O_2), \quad \lambda_U \approx U / |t| \geq 0 \\
O_1 &= \psi_1(z) \bar{\psi}_1(\bar{z}) \psi_2(z) \bar{\psi}_2(\bar{z}) + \psi_1(z) \bar{\psi}_1(\bar{z}) \psi_3(z) \bar{\psi}_3(\bar{z}) \\
&\quad + \psi_2(z) \bar{\psi}_2(\bar{z}) \psi_3(z) \bar{\psi}_3(\bar{z}) \\
O_2 &= \psi_0(z) \bar{\psi}_0(\bar{z}) (\psi_1(z) \bar{\psi}_1(\bar{z}) + \psi_2(z) \bar{\psi}_2(\bar{z}) + \psi_3(z) \bar{\psi}_3(\bar{z}))
\end{aligned}$$

The zigzag interaction is, in the fermionic language:

$$V_{ZigZag} = \lambda_Z \left[(O_1 - O_2) + \sum_{i=0}^3 (T^{(i)}(z) + \bar{T}^{(i)}(\bar{z})) \right], \quad \lambda_Z \approx J_{\perp}^Z / |t| \geq 0$$

Such an interaction is marginally relevant and contains a non scalar term, whose effect is simply that of renormalizing the fields and the velocities. After renormalization, the whole effect of the two terms V_U and V_{ZigZag} can be expressed as a marginal interaction:

$$VV_{ZigZag} = \lambda_+^0 (O_1 + O_2) + \lambda_-^0 (O_1 - O_2), \quad \lambda_{\pm}^0 = \frac{1}{2(1 + \pi\lambda_Z)} \left\{ \mp \frac{\lambda_Z + \lambda_U}{1 - 3\pi\lambda_Z} + \frac{\lambda_Z - \lambda_U}{1 + \pi\lambda_Z} \right\}$$

The following RG equations hold:

$$\frac{d\lambda_{\pm}^0}{d \ln L} = 8\pi (\lambda_{\pm}^0)^2$$



Under the flow λ_+^0 renormalizes to zero while λ_-^0 increases and that results in a dynamical length scale $\xi \approx \exp(1/\lambda_-^0)$. The Z_2 symmetry of L_0 and V_{ZigZag} spontaneously breaks, a mass scale appears dynamically and provides a non vanishing mass for the four fermions:

$$m_i \approx v_i \xi^{-1}, i = 1, 2, 3; \quad m_0 \approx v_0 \xi^{-1}$$

$$m_1 = m_2 = m_3 = m > 0; \quad m_0 < -m$$

It is not possible to extract trajectories in the RG flow characterized by a vanishing mass in the $c=1/2$ or $c=3/2$ sector respectively. So, we need to introduce the 4-dimer perturbation:

$$V_{4-Dimer} = \lambda_\kappa (O_1 + O_2) = \lambda_\kappa \sum_{i \neq j=0}^3 \psi_j(z) \bar{\psi}_j(\bar{z}) \psi_i(z) \bar{\psi}_i(\bar{z}), \quad \lambda_\kappa \approx J_\kappa$$

The whole perturbation is:

$$V_{Tot} \equiv V_{ZigZag} + V_{4-Dimer} = \lambda_+ (O_1 + O_2) + \lambda_- (O_1 - O_2)$$

$$\lambda_+ = \lambda_+^0 + \lambda_\kappa, \quad \lambda_- = \lambda_-^0$$



It is now possible to define a path in the RG flow characterized by a vanishing singlet mass, i.e. $m_t \neq 0$, $m_s = 0$, as shown by rewriting V_{Tot} as:

$$V_{Tot} = \Lambda_1 O_1 + \Lambda_2 O_2$$
$$\Lambda_1 = \lambda_+ + \lambda_-, \quad \Lambda_2 = \lambda_+ - \lambda_-$$

A vanishing singlet mass m_s can be obtained by requiring the marginality of the operator O_2 along the RG flow. This selects the condition $\lambda_+ = \lambda_-$ which makes Λ_2 to vanish. The triplet mass m_t is dynamically generated and reads as (F is the momentum cutoff and v_t is the spin triplet velocity):

$$m_t \approx \pm v_t F \exp(-1/8\pi\lambda_+)$$



The 4-Dimer interaction allows us to describe a RG trajectory flowing from the $c=2$ UV fixed point (TM) toward the Ising I , the $c=1/2$ IR fixed point, as a result of the dynamical generation of the mass m_t and the consequent decoupling of $su(2)_2$.