

Convegno Informale di Fisica Teorica  
Sestri Levante  
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# Rayleigh–Taylor instability and phase-field model

— surface-tension effects and numerical description —

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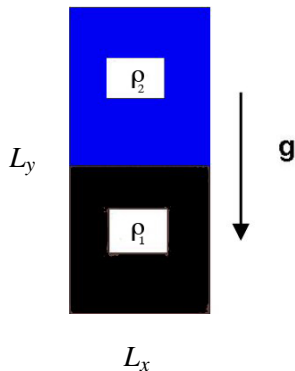
<sup>3</sup>Department of Mathematics and Statistics, University of Helsinki



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# The numerical framework



- **immiscible**, homogeneous fluids (density  $\rho_2 > \rho_1$ ), in gravitational field  $g$
- **two-dimensional** case



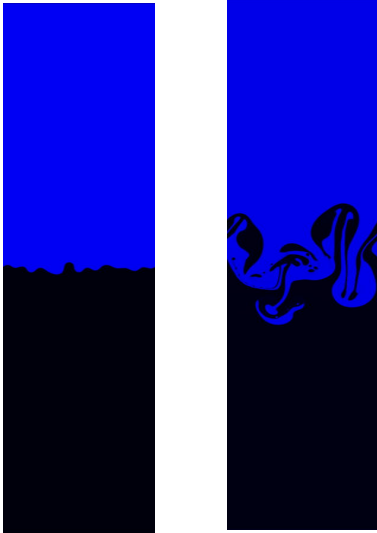
# Time evolution: a qualitative idea



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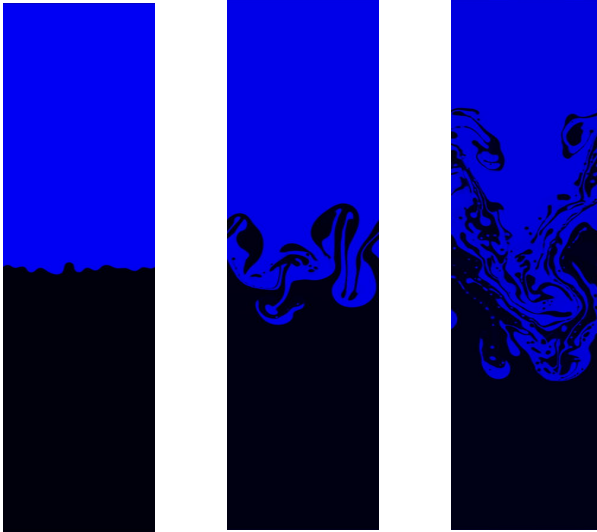
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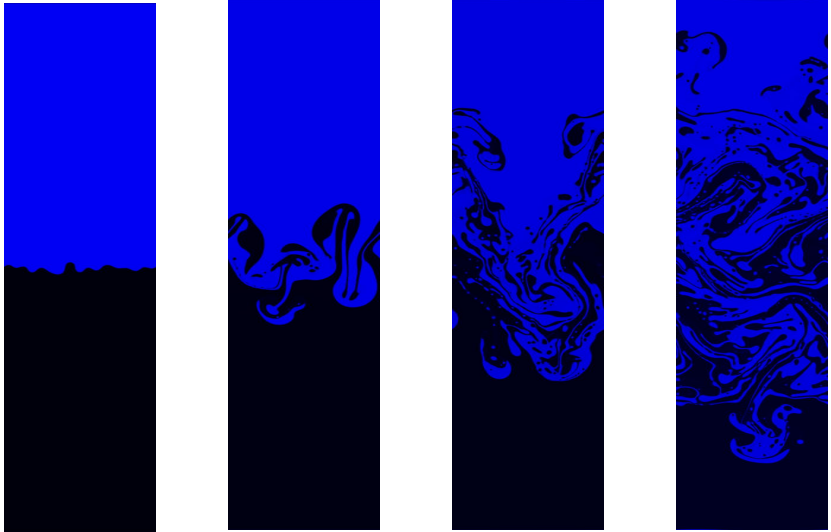
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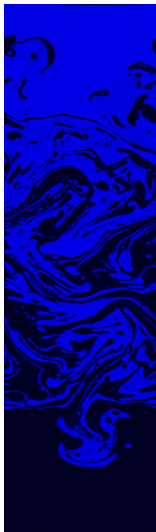
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# Time evolution: a qualitative idea



# Numerical description: difficulties



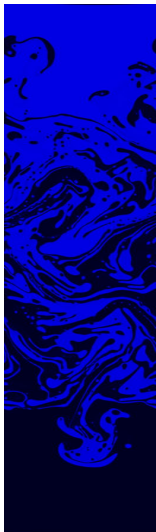
- sharp interface
- strong deformation of interface



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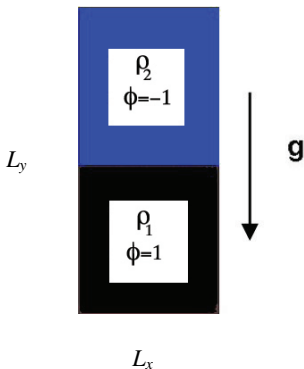


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# Numerical description: phase-field model (1)



- sharp interface  $\Rightarrow$  diffuse interface
- phase field  $\phi(x)$
- Ginzburg–Landau free energy

$$\mathcal{F}[\phi] = \int_{\Omega} \left( \frac{\Lambda}{2} |\nabla \phi(\mathbf{x})|^2 + \frac{\Lambda}{4\epsilon^2} (\phi^2 - 1)^2 \right) dx$$

J.W. CAHN & J.E. HILLIARD *J. Chem. Phys.* **28** (1958)  
 A.J. BRAY *Advances in Physics* **51**(2) (2002)



# Numerical description: phase-field model (2)

- equilibrium:  $\frac{\delta \mathcal{F}}{\delta \phi} = 0$ 
  - $\phi(y) = \pm \tanh\left(\frac{y}{\sqrt{2}\varepsilon}\right)$
  - $\sigma = \frac{2\sqrt{2}}{3} \frac{\Lambda}{\varepsilon}$
- out of equilibrium :  $\frac{\delta \mathcal{F}}{\delta \phi} \neq 0$

- sharp-interface limit:

$$\varepsilon, \Lambda \rightarrow 0 \quad \text{with } \sigma \sim \frac{\Lambda}{\varepsilon} \text{ fixed}$$

$$\gamma \rightarrow 0$$



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# Linear stage: growth rate

$$t = 0: \eta(x, 0) = \eta_0 \sin(kx), \frac{\eta_0}{\lambda} \ll 1 \text{ and } \frac{\eta_0}{\varepsilon} \gg 1$$



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phase-field formulation  $\Rightarrow$   
perturbation amplitude:

$$\eta(t) = \eta_0 \cosh(\alpha t)$$

with:

$$\alpha^2 = \mathcal{A}gk - \frac{\sigma}{\rho_2 + \rho_1} k^3$$

$$\left( \mathcal{A} = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \right)$$

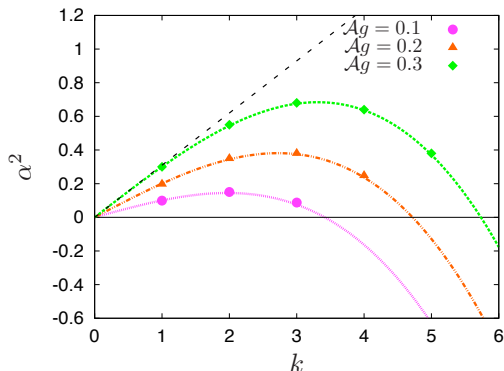
S. CHANDRASEKHAR Hydrodynamics and Hydromagnetic Stability *New York: Dover* (1961)





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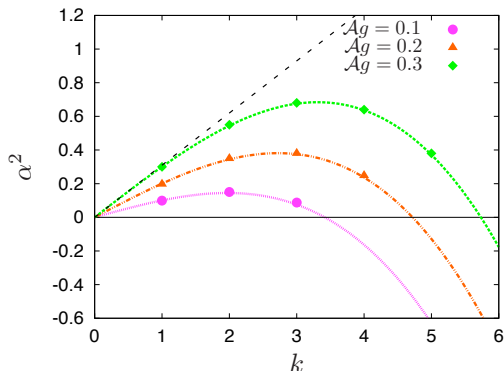
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S. CHANDRASEKHAR Hydrodynamics and Hydromagnetic Stability New York: Dover (1961)

$\frac{\eta(t)}{\lambda} \sim 0.1 \Rightarrow$  deviations from the linear theory are observed ...

[D.H. SHARP *Physica D* 12 (1984)]



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# Weakly-nonlinear stage

$$U = \sqrt{\frac{2}{3} \mathcal{A} \frac{g}{k} - \frac{2}{9} \frac{\sigma}{\rho_2 + \rho_1} k}$$

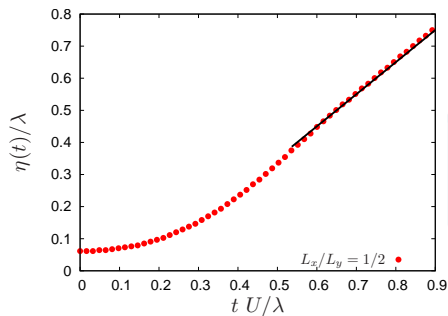
Y.N. YOUNG & F.E. HAM *J. Turbul.* **71(7)** (2006)



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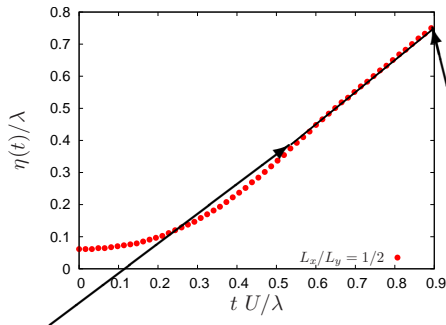


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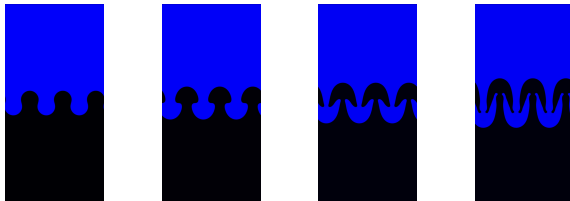


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experiments:

J.T. WADDELL ET AL.  
*Phys. Fluids* **13(5)** (2001)



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# Conclusions: . . .

## 2D Rayleigh–Taylor instability

- phase-field model
- effects of  $\sigma$  on linear and weakly nonlinear stages

⇒ full details: A. CELANI, A. MAZZINO, P. MURATORE-GINANNESCHI AND L.V.  
*J. Fluid Mech.* **622**, 115-134 (2009)

## . . . Perspective

- turbulent stage  
⇒ phenomenological theory: M. CHERTKOV ET AL.  
*Phys. Rev. E* **71**, 055301(R) (2005)



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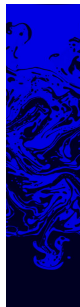
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# Outline

- 1 Phase-field model
- 2 Some of our numerical results





# Outline of the Appendix

- 3 Phenomenology
  - Time evolution: immiscible configuration
  - Time evolution: miscible configuration
  
- 4 Other results
  - Linear phase
  - Weakly nonlinear stage



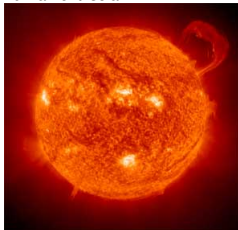
# Where? Some examples

supernovae



ZINGALE ET AL. *J. Phys.: Conf. Ser.* **16**  
(2005)

brillamenti solari



ISOBE ET AL. *Nature* **434**  
(2005)

nuvole mammatus



SCHULTZ ET AL. *J. Atmos. Sci.* **63** (2006)

...



# Surface tension effect - 1

- Linear stage:

$$\eta(t) \sim e^{\alpha(k, Ag, \sigma, \nu)t}$$

S. CHANDRASEKHAR *Hydrodynamics and Hydromagnetic Stability*  
*New York: Dover* (1961)

- Weakly nonlinear stage:

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U. ALON ET AL. *PRL* **74**(4) (1995)

Y.N. YOUNG & F.E. HAM *J. Turbul.* **71**(7) (2006)

- Turbulent stage:

$$L(t) \sim Agt^2$$

$$\ell(t) \sim t^{2/11} \text{ capillary scale}$$

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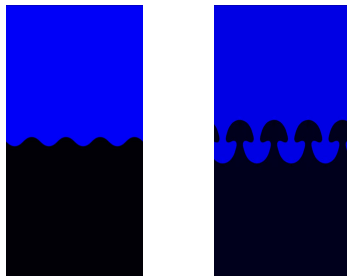
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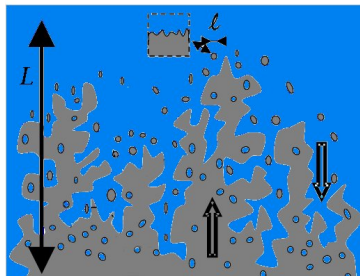
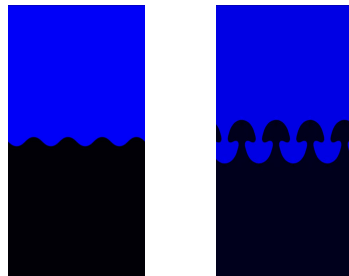
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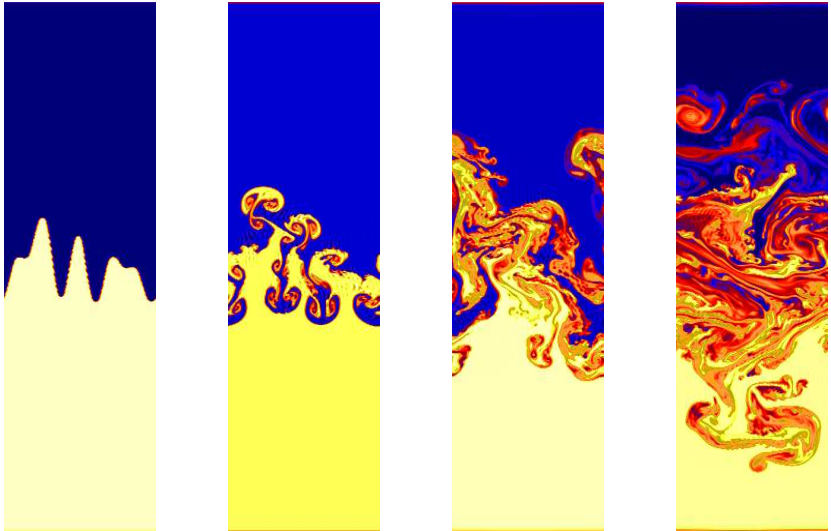
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*Phys. Rev. E* **71** (2005)



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# Miscible configuration: a qualitative idea



# Growth rate - 1

- free energy:  $\mathcal{F}[\phi] = \int_{\Omega} \frac{\Lambda}{2} |\nabla \phi(\mathbf{x})|^2 + \frac{\Lambda}{4\varepsilon^2} (\phi^2 - 1)^2 dx$
- $h(x, t)$  initial perturbation  $\Rightarrow \phi = f\left(\frac{y-h(x, t)}{\varepsilon}\right)$

- locally the interface is in equilibrium:  $f'' = V'(f) \Rightarrow$

$$\mu = -\Lambda \frac{\partial^2 f}{\partial x^2} = \frac{\Lambda}{\varepsilon} \left[ f' \frac{\partial^2 h}{\partial x^2} - \frac{f''}{\varepsilon} \left( \frac{\partial h}{\partial x} \right)^2 \right]$$

- by linearizing Navier–Stokes equation for small interface velocity ( $\nu = 0$ ):

$$\rho_o \partial_t v = -\partial_y p - \phi \partial_y \mu - \mathcal{A} g \rho_o \phi$$

- integration along the vertical direction:

$$\rho_o \partial_t q_y = \sigma \frac{\partial^2 h}{\partial x^2} + 2\mathcal{A} g \rho_o h$$

where  $q_y := \lim_{L \uparrow \infty} \int_{-L}^L v dy$

- vertical velocity at interface:  $\partial_t h = v(x, h(x, t), t) \equiv v^{(int)}(x, t)$



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# Growth rate - 2

- ... • relation between  $v^{int}$  and  $q_y$
- potential flow:  $\mathbf{v} = \partial \psi$   $\partial^2 \psi = 0 \Rightarrow$

$$\hat{v}^{(int)} = \frac{k \hat{q}_y}{2}$$

- in  $k$  space:  $\partial_t \hat{h} = \frac{k \hat{q}_y}{2}$   $\rho_o \partial_t \hat{q}_y = (-\sigma k^2 + 2Ag\rho_o) \hat{h}$

## Growth rate $\alpha$

$$\partial_t^2 \hat{h} - \alpha^2 \hat{h} = 0$$

where:

$$\alpha^2 = Agk - \frac{\sigma}{2\rho_o} k^3$$

S. CHANDRASEKHAR Hydrodynamics and Hydromagnetic Stability *New York: Dover* (1961)

...



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## Growth rate - 3

- ... • the initial condition:  $\hat{h}(k, 0) = \delta(k - \bar{k}), \quad \partial_t \hat{h} = 0 \quad \text{at} \quad t = 0$
- the solution:  $\hat{h}(\bar{k}, t) = \hat{h}(\bar{k}, 0) \cosh(\alpha t)$
  - $\Rightarrow \hat{v}^{(int)}(\bar{k}, t) = -\hat{h}(\bar{k}, 0) \sinh(\alpha t) \Rightarrow \hat{\psi}(\bar{k}, t) = \dots \Rightarrow$
  - in  $\mathbb{R}$  space:

$$h(x, t) = h_0 \cos(\bar{k}x) \cosh(\alpha t)$$

$$v^\uparrow(x, y, t) \equiv v(x, y, t) = \cos(\bar{k}x) e^{-\bar{k}y} h_0 \alpha \sinh(\alpha t)$$

$$u^\uparrow(x, y, t) \equiv u(x, y, t) = -\sin(\bar{k}x) e^{-\bar{k}y} h_0 \alpha \sinh(\alpha t)$$

for  $y > 0$ , and:

$$v^\downarrow(x, y, t) \equiv v(x, y, t) = \cos(\bar{k}x) e^{+\bar{k}y} h_0 \alpha \sinh(\alpha t)$$

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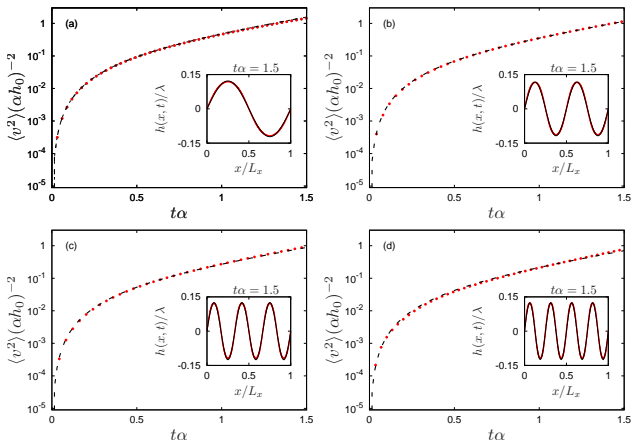


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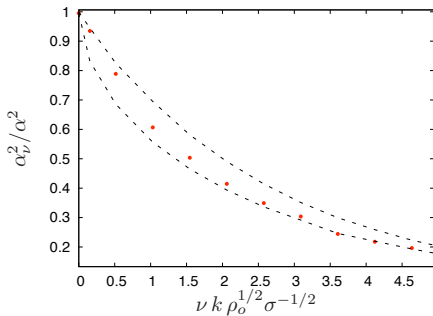


## Temporal evolution

$$\langle v^2 \rangle = \frac{1}{2a_y k} [-e^{-2a_y} + 1] \alpha^2 h_0^2 \sinh^2(\alpha t)$$



# Viscosity effect



$$\alpha_\nu \equiv \alpha = \alpha(k, Ag, \sigma, \nu)$$

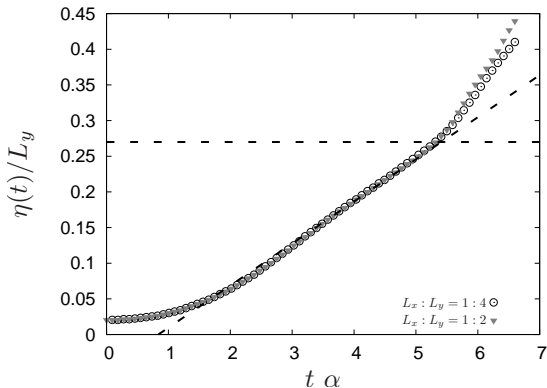
R. MENIKOFF ET AL. *Phys. Fluids* **20** (1977)



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## Effect of finite size



$$\alpha = \sqrt{Agk - \frac{\sigma}{\rho_1 + \rho_2} k^3}$$

